Homework (Bognar)
Introduction to Mathematical Statistics II (STAT:3101)

1. Suppose $X_1, \ldots, X_n$ is a random sample from a $Pois(\theta)$ distribution with pmf

$$f(x|\theta) = \frac{e^{-\theta}\theta^x}{x!}$$

for $x = 0, 1, 2, \ldots$ and $\theta > 0$. Suppose the prior distribution on $\Theta \sim Gamma(\alpha, \beta)$ with pdf

$$h(\theta) = \frac{1}{\Gamma(\alpha)\beta^\alpha}\theta^{\alpha-1}e^{-\theta/\beta}$$

(a) Find the posterior distribution of Θ given $X_1 = x_1, \ldots, X_n = x_n$, i.e. find $k(\theta|x_1, \ldots, x_n)$. *Hint: It is a gamma distribution.*

(b) Find the posterior mean of Θ given $X_1 = x_1, \ldots, X_n = x_n$, i.e. find $E(\Theta|X_1 = x_1, ..., X_n = x_n)$.

(c) Show that the estimate in (1b) is a weighted average of the prior mean $\alpha\beta$ and the MLE $\bar{x}$ with respective weights $1/(n\beta + 1)$ and $n\beta/(n\beta + 1)$.

(d) Suppose a researcher lets $\alpha = \beta = 2$. If she observes $X_1 = 6$, $X_2 = 3$, $X_3 = 2$, $X_4 = 4$, what is the Bayesian estimate of θ (using the posterior mean)?

(e) Suppose a researcher lets $\alpha = \beta = 3$. If she observes $X_1 = 6$, $X_2 = 3$, $X_3 = 2$, $X_4 = 4$, what is the Bayesian estimate of θ (using the posterior mean)?

(f) Are you surprised that the estimate in (1e) is larger than in (1d)? Why?

(g) Looking at your answer in (1c), which has more influence on the posterior mean as the sample size increases, the prior mean $\alpha\beta$ or the MLE $\bar{x}$?