NOTES ON ERDÖS' CONJECTURE

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Abstract  Let $X_n, n \geq 1$, be a sequence of independent random variables satisfying $P(X_n = 0) = 1 - P(X_n = a_n) = 1 - 1/p_n$, where $a_n, n \geq 1$, is a sequence of real numbers, and $p_n$ is the $n$th prime set $F_N(x) = P\left( \sum_{n=1}^{N} X_n \leq x \right)$. The authors investigate a conjecture of Erdős in probabilistic number theory and show that in order for the sequence $F_N$ to be weakly convergent, it is both sufficient and necessary that there exist three numbers $x_2$ and $x_1 < x_2$ such that $\limsup_{N \to \infty} (F_N(x_2) - F_N(x_1)) > 0$ holds, and $L_0 = \lim_{N \to \infty} F_N(x_0)$ exists. Moreover, the authors point out that they can also obtain the same result in the weakened case of $\liminf_{N \to \infty} P(X_n = 0) > 0$.

Key words  Erdős' conjecture, additive arithmetic function, sums of independent random variables, essential convergence, weak convergence

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1 Introduction

A function $f : N \to \mathbb{R}$ is called additive if $f(mn) = f(m) + f(n)$ for any coprime integers $m$ and $n$. Given an additive function $f$, we can define, for each $N \in \mathbb{N}$, a frequency function

$$F_N(x) = u_N(n : f(n) \leq x) \Delta \frac{1}{N} \cdot \# \{n \leq N : f(n) \leq x\}.$$ 

Erdős conjectured that in order for the sequence $F_N$ to be (weakly) convergent, it is sufficient that there exist two numbers $x_1 < x_2$ such that

$$P_0 = \lim_{N \to \infty} (F_N(x_2) - F_N(x_1))$$ exists and is positive. \hfill (1.1)

This old conjecture remains open to date.

As noted in Elliott\(^{[1]}(p330-331)\), standard techniques from probabilistic number theory show that this conjecture is equivalent to the following purely probabilistic statement.

Let $a_n, n \geq 1$, be a sequence of real numbers and let $X_n, n \geq 1$, be a sequence of independent random variables satisfying

$$P(X_n = a_n) = \frac{1}{p_n}, \quad P(X_n = 0) = 1 - \frac{1}{p_n}, \quad \hfill (1.2)$$

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where \( p_n \) is the \( n \)-th prime number. In order for the distributions

\[
F_N(x) = P\left( \sum_{n=1}^{N} X_n \leq x \right)
\]

(1.3)

to be weakly convergent, it is sufficient that (1.1) holds.

Up to the present only some partial results have been obtained. In particular, Babu [2] (Chapter 4) showed that the conclusion of Erdős' conjecture holds if Erdős' condition (1.1) is replaced by the stronger condition

\[
f(z) = \lim_{N \to \infty} (F_N(z) - F_N(z_1)) \text{ exists for } z_1 \leq z \leq z_2 \text{ and is not a linear function of } z; \quad (1.4)
\]

Hildebrand [3] showed that the conclusion of the conjecture holds if (1.1) is replaced by the following stronger condition

\[
L_i = \lim_{N \to \infty} F_N(z_i) \text{ exists for } i = 1, 2 \text{ and } L_1 \neq L_2. \quad (1.5)
\]

Based upon Elliott's statement, the further research on this conjecture will be done. Throughout this paper, we assume that \( X_n, n \geq 1 \), is always a sequence of independent random variables and that \( F_N \) denote the distributions (1.3). Our principal results are as follows.

**Theorem 1** Let \( X_n, n \geq 1 \), satisfy (1.2), where \( a_n, n \geq 1 \), have the same signs. In order for the sequence \( F_N \) to be weakly convergent, it is both sufficient and necessary that there exist two numbers \( z_1 < z_2 \) such that

\[
\limsup_{N \to \infty} (F_N(z_2) - F_N(z_1)) > 0. \quad (1.6)
\]

**Theorem 2** Let \( X_n, n \geq 1 \), satisfy(1.2), then in order for the sequence \( F_N \) to be weakly convergent, it is both sufficient and necessary that (1.6) holds, and there exists some number \( z_0 \) such that

\[
L_0 = \lim_{N \to \infty} F_N(z_0) \text{ exists.} \quad (1.7)
\]

Clearly the condition of Theorem 2 is weaker than Hildebrand's condition (1.5). This result can be generalized in the following way.

**Theorem 3** Let \( X_n, n \geq 1 \), satisfy

\[
\liminf_{n \to \infty} P(X_n = 0) > 0, \quad (1.8)
\]

then in order for the sequence \( F_N \) to be weakly convergent, it is both sufficient and necessary that (1.6), (1.7) hold.

2 Preliminaries

As is known, \( \sum_{n=1}^{\infty} X_n \) is said to be essentially convergent almost surely (a.s.) if there exist constants \( b_n, n \geq 1 \), such that

\[
\sum_{n=1}^{\infty} (X_n - b_n) \quad (2.1)
\]
a.s. converges. We shall need the following related results which can be found in Stout[6].

Lemma 1 \[ \sum_{n=1}^{\infty} X_n \] essentially converges a.s. if and only if

\[ \sum_{n=1}^{\infty} P(|X_n - \mu(X_n)| \geq c) < \infty, \] (2.2)

\[ \sum_{n=1}^{\infty} \text{Var}(X_n - \mu(X_n)) < \infty \] (2.3)

hold for some constant \( c > 0 \), where, \( \mu(X) \) is a median of a random variable \( X \), and \( X^c \) denotes \( X \) truncated at \( \pm c \). Moreover, if (2.2), (2.3) hold for some constant \( c > 0 \) they hold for all \( c > 0 \), and the series (2.1) a.s. converges for

\[ b_n = \mu(X_n) + E((X_n - \mu(X_n))I(|X_n - \mu(X_n)| < 1)). \] (2.4)

Lemma 2

\[ \limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} X_n < K > 0 \] (2.5)

for some constant \( K > 0 \) implies that \( \sum_{n=1}^{\infty} X_n \) essentially converges a.s.

We define the range of a random variable \( X \) as the set

\[ R(X) = \{ x \in \mathbb{R} : P(|X - x| \leq \epsilon) > 0, \text{ for every } \epsilon > 0 \}. \]

It can be easily seen that \( R(X) \) is always a nonempty closed set satisfying

\[ R(X + l) = \{ x + l : x \in R(X) \} \] (2.6)

for any constant \( l \). We shall need the next lemma, which can be naturally deduced from the proof of Lemma 2 in Hildebrandt[5].

Lemma 3 Let \( \sum_{n=1}^{\infty} X_n \) be a.s. convergent and let \( F \) denote its limiting distribution. Suppose that for every \( \epsilon > 0 \) and \( n > n_0(\epsilon) \), there exist numbers \( C_n^{(i)} = C_n^{(i)}(\epsilon) \in R(X_n) \) with \( |C_n^{(i)}| < \epsilon \), such that \( \sum_{n=n_0+1}^{\infty} C_n^{(i)} = (-1)^i \infty, i = 1, 2 \), then \( F \) is strictly increasing.

3 Proof of Theorem 1

Before giving the proof, from (1.6) we deduce a preliminary proposition which will be used for several times. From (1.6), the hypothesis (2.5) of Lemma 2 is satisfied for any \( K > \max\{|x_1|, |x_2|\} \), and it follows that \( \sum_{n=1}^{\infty} X_n \) essentially converges a.s. Then according to Lemma 1, (2.2), (2.3) hold for any \( c > 0 \), and the series (2.1) a.s. converges, for \( b_n \) given by (2.4). We write

\[ \begin{cases} 
Y_n = X_n - b_n, Z_n = X_n - \mu(X_n), A_N = \sum_{n=1}^{N} b_n, \\
Y = \sum_{n=1}^{\infty} Y_n, G_N(z) = P \left( \sum_{n=1}^{N} Y_n \leq z \right), G(z) = P(Y \leq z). 
\end{cases} \] (3.1)
Clearly
\[ F_N(x) = G_N(x - A_N). \] (3.2)

By a well-known result, so-called 'equivalence theorem', which says, for series of independent random variables, the weak convergence of the distributions of its partial sums is equivalent to its a.s. convergence, together with (2.1), we come to learn that (1) the weak convergence of \( F_N \), (2) the a.s. convergence of \( \sum_{n=1}^{\infty} X_n \) and (3) the convergence of \( A_N \) are equivalent.

Moreover, by (1.6) we can assert that
\[ A_N \text{ has at least one finite limiting point.} \] (3.3)

In fact, if it isn't so, we have \( A_N \to \infty \) as \( N \to \infty \), then by (3.2),
\[ \limsup_{N \to \infty} (F_N(x_2) - F_N(x_1)) = \limsup_{N \to \infty} (G_N(x_2 - A_N) - G_N(x_1 - A_N)) = 0, \]
which contradicts our assumption (1.6). By (3.3), therefore the convergence of \( A_N \) may be guaranteed if one of the following two conditions
\[ (1) \sum^+ b_n < \infty, \quad (2) \sum^- b_n > -\infty \] (3.4)
holds, where, \( \sum^+ \) and \( \sum^- \) denote that the summations are confined to those positive and negative terms respectively.

Combining this with above, we have

**Lemma 4** Let \( X_n, n \geq 1 \), satisfy (1.6), then \( F_N \) weakly converges if and only if \( A_N \) converges; and \( F_N \) weakly converges if one of the two conditions in (3.4) holds.

By Lemma 4, we may assume without loss of generality that
\[ \sum^- b_n = -\infty \text{ and } \sum^+ b_n = \infty. \] (3.5)

**Proof of Theorem 1** Trivially. The weak convergence of \( F_N \) implies (1.6), then we only need to prove the 'sufficient' assertion. For all \( n \geq 1 \), by (1.2),
\[ \mu(X_n) = 0, \] (3.6)
then by (2.4),
\[ b_n = \frac{\alpha_n}{p_n} I(|\alpha_n| < 1), \] (3.7)
hence clearly at least one of the conditions in (3.4) holds since \( a_n, n \geq 1 \), have the same signs.

Finally, by Lemma 4 we obtain the weak convergence of \( F_N \).

This completes the proof of Theorem 1.

**Remark 1** If \( a_n, n \geq 1 \), are not the same signed, the conclusion of Theorem 1 does not always hold. As further evidence we give the following counterexample.

**Example 1** we know
\[ \sum_{n=1}^{\infty} \frac{1}{p_n \log \log p_n} = \infty; \] (3.8)
\[ \sum_{n=1}^{\infty} \frac{1}{p_n (\log \log p_n)^2} < \infty. \] (3.9)
Let $C_0 > 0$ bound the sequence $1/p_n \log \log p_n$, $n \geq 1$, and let $C > C_0$ be arbitrarily fixed. By (3.8), there exists a subsequence $n_k, k \geq 1$ such that
\[
\left| \left( \sum_{n=1}^{n_1} - \sum_{n=n_{k-1}+1}^{n_k} + \cdots + (-1)^{k-1} \sum_{n=n_{k-1}+1}^{n_k} \right) \frac{1}{p_n \log \log p_n} \right| \leq C
\]
and
\[
\left| \left( \sum_{n=1}^{n_1} - \sum_{n=n_{k-1}+1}^{n_k} + \cdots + (-1)^{k-1} \sum_{n=n_{k-1}+1}^{n_k} \right) \frac{1}{p_n \log \log p_n} \right| > C,
\]
for all $k \geq 1$, where $n_0 = 0$. Set $a_n = (-1)^{\tau_n} / \log \log p_n$, where, $\tau_n = k - 1$ as $n_{k-1} < n \leq n_k$, for all $n \geq 1$. We construct a sequence of independent random variables $X_n, n \geq 1$, satisfying (1.2). As seen, $\sum_{n=1}^{\infty} EX_n = \sum_{n=1}^{\infty} E X_n = \sum_{n=1}^{\infty} a_n$ is a divergent series of which the partial sums $A_N = \sum_{n=1}^{N} a_n$ are bounded by the constant $C$ and furthermore, the set of all limiting points of $A_N$ equals the interval $[-C, C]$. By Kolmogorov's three series theorem, $\sum_{n=1}^{\infty} X_n$ a.s. diverges, equivalently, $F_N$ does not weakly converge.

On the other hand, let $U > 0$, set $z_i = (-1)^i (U + 1) C_i, i = 1, 2$, by Chebyshev's inequality, we have
\[
F_N(z_2) - F_N(z_1) \geq P \left( \sum_{n=1}^{N} X_n - E \sum_{n=1}^{N} X_n \right) < UC
\]
\[
\geq 1 - \frac{1}{(UC)^2} \sum_{n=1}^{\infty} \frac{1}{p_n (\log \log p_n)^2} \left( 1 - \frac{1}{p_n} \right),
\]
by (3.9) and let $U$ be sufficiently large such that the right-hand of above is positive. So that (1.5) holds.

Indeed, a slightly better result can be obtained. By Lemma 5 in Section 4, we shall see almost at once that the distribution $G$ in (3.1) corresponding to this example is continuous and strictly increasing. We can obtain that $G_N(x)$ converges to $G$ uniformly in $x \in \mathbb{R}$ by using a theorem of Petrov[8], which states that if the sequence of distributions converges to a continuous distribution, the convergence is uniform. Hence for any $z_1 < z_2$, by (3.2), we have
\[
\limsup_{N \to \infty} (F_N(z_2) - F_N(z_1)) = \limsup_{N \to \infty} (G_N(z_2 - A_N) - G_N(z_1 - A_N))
\]
\[
= \sup_{t \in [-C, C]} (G(z_2 - t) - G(z_1 - t)) > 0.
\]

**Remark 2** While we cannot decide whether the Erdős condition (1.1) is already sufficient in order for $F_N$ to be weakly convergent, we show in Example 1 that it cannot be weakened to (1.6), or to $\liminf_{N \to \infty} (F_N(z_2) - F_N(z_1)) > 0$. We also show in Example 1 that even under the case of (1.2) there exists an a.s. essentially convergent series of independent random variables which is not a.s. convergent.
4 Proof of Theorem 2

By (1.2), (2.4), we see (3.6), (3.7) hold. As we know, \( \sum_{n=1}^{\infty} \frac{1}{p_n} = \infty \), If \( \sum_{n=1}^{\infty} \frac{1}{p_n} < \infty, \sum_{n=1}^{\infty} b_n = \sum_{(\alpha_n \in P^c, |\alpha_n| < 1)} a_n \) absolutely converges, then by Lemma 4 we obtain the weak convergence of \( F_N \). Therefore in this section we may assume without loss of generality that

\[
\sum_{(\alpha_n \neq 0)} \frac{1}{p_n} = \infty.
\]  

(4.1)

It is of course already sufficient to prove the Theorem 2 under our hypotheses (3.5), (4.1). We first give a preliminary proposition which is of independent use.

Lemma 5 Let \( X_{\xi}, n \geq 1 \), satisfy (1.2), (1.6), (3.5), (4.1), then the distribution \( G \) in (3.1) is continuous and strictly increasing.

Proof Note (1.2), (4.1), the continuity of \( G \) is a consequence of a theorem of Lévy[6] which states that an a.s. convergent series \( \sum_{n=1}^{\infty} \xi_n \) of independent random variables has a continuous limiting distribution if and only if, for any sequence \( d_n, n \geq 1 \), \( \sum_{n=1}^{\infty} P(\xi_n \neq d_n) = \infty \).

For proving the strict monotonicity of \( G \), we use Lemma 3. Let \( \varepsilon, 0 < \varepsilon < 1 \), be arbitrarily fixed, set \( C_n^{(1)} = a_n I(-\varepsilon/2 < a_n < 0) - b_n, C_n^{(2)} = a_n I(0 < a_n < \varepsilon/2) - b_n, n \geq 1 \). By (1.2), (2.6), (3.1) we see \( C_n^{(i)} \in R(Y_\xi), i = 1, 2, n \geq 1 \). By (3.7),

\[
\lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{a_n}{p_n} I(|a_n| < 1) = 0,
\]

(4.2)

thus there exists \( n_1 = n_1(\varepsilon) \) such that for all \( n > n_1, |b_n| < \varepsilon/2 \) and therefore \( |C_n^{(i)}| < \varepsilon, i = 1, 2 \). Moreover, by (2.2), (3.5), (3.6), (3.7), we have

\[
\sum_{n=n_1+1}^{\infty} C_n^{(1)} = \sum_{n=n_1+1}^{\infty} (a_n(1 - \frac{1}{p_n})I(-\varepsilon/2 < a_n < 0) - \frac{a_n}{p_n} I(-1 < a_n \leq -\varepsilon/2)
\]

\[
- \frac{a_n}{p_n} I(0 < a_n < 1)) \\
\leq \sum_{n=n_1+1}^{\infty} \frac{a_n}{p_n} - \sum_{n=n_1+1}^{\infty} \frac{a_n}{p_n} = \sum_{n=n_1+1}^{\infty} \frac{a_n}{p_n} \to \infty.
\]

Similarly, \( \sum_{n=n_1+1}^{\infty} C_n^{(2)} = \infty \). Now using Lemma 3 yields the desired result.

Proof of Theorem 2 Trivially, the weak convergence of \( F_N \) implies (1.6), (1.7), then we only need to prove the 'sufficient' assertion. Now we may readily obtain the convergence of \( A_N \). Actually, if \( A_N \) diverges, there exist two subsequences \( A_{N_n} \) and \( A_{N^*} \) such that

\[
\lim_{N \to \infty} A_{N_n} = A_n \overset{\Delta}{=} \liminf_{N \to \infty} A_N < \limsup_{N \to \infty} A_N \overset{\Delta}{=} A^* = \lim_{N \to \infty} A_{N^*}.
\]

(4.3)

Since \( G \) is strictly increasing and \( G_N \) converges uniformly to \( G \) (because of the continuity of \( G \) and the theorem of Petrov which we mentioned in Remark 1), using (1.7), (3.2), (4.3), we have reached a contradiction by deducing

\[
L_0 = \lim_{N \to \infty} F_N(x_0) = \lim_{N \to \infty} G_N(x_0 - A_N^*) = G(x_0 - A^*) < G(x_0 - A_*) = L_0.
\]

(4.4)
Finally by Lemma 4, we obtain the weak convergence of $F_N$.

This completes the proof of Theorem 2.

Remark 3 Suppose (1.1), (1.2) are valid. In the similar way of the proof of (3.3), we can easily obtain from (1.1) that $A_N$ must be bounded. Thus, if $A_N$ diverges, say

\[ -\infty < A_* \overset{\Delta}{=} \liminf_{N \to \infty} A_N < \limsup_{N \to \infty} A_N \overset{\Delta}{=} A^* < \infty, \]

by (1.2), we have (3.6), (3.7), (4.2) in turn, so that the set of all limiting points of $A_N$ equals the interval $[A_*, A^*]$. Then under our hypotheses (3.5), (4.1), by Lemma 5, $G$ is continuous and strictly increasing. Moreover, we can obtain that $G$ satisfies

\[ G(x_2-t) - G(x_1-t) \equiv P_0 > 0 \quad \text{for every} \quad t \in [A_*, A^*]. \quad (4.5) \]

Suppose Babu’s condition (1.4) holds, combining (4.5) with (1.4) we can reach a contradiction if $A_N$ diverges. Then by Lemma 4 we obtain the result of Babu.

Remark 4 In Theorem 2, the condition (1.6) cannot be omitted even if (1.7) is strengthened to $0 < L_0 < 1$. As further evidence we give the following counterexample.

Example 2 Let $d_n, n \geq 1$, be a sequence of positive numbers satisfying

\[ d_N > \sum_{n=1}^{N-1} d_n + N \quad \text{for all} \quad N \geq 2. \]

We construct a sequence of independent random variables $X_n, n \geq 1$, satisfying (1.2), where $a_n = (-1)^n d_n$. By Kolmogorov’s three series theorem, $\sum_{n=1}^{\infty} X_n$ a.s. diverges, equivalently, $F_N$ does not weakly converge.

On the other hand, set a random variable $\tilde{N} = \max\{n \leq N : X_n = a_n\}$, by (1.2),

\[ P(\tilde{N} = m) = \frac{1}{P_m} \prod_{n=m+1}^{N} \left(1 - \frac{1}{P_n}\right) \quad \text{for} \quad 0 \leq m \leq N, \]

\[ \text{Sign}(\sum_{n=1}^{N} X_n) = (-1)^{\tilde{N}}, \quad (4.7) \]

where Sign$(x)$ denotes the sign function. For arbitrarily fixed $M > 0$ and every $N > M + 1$, by (4.7) we have

\[ \left( \sum_{n=1}^{N} X_n \leq -M \right) = (\tilde{N} \text{ is odd}) \setminus (\tilde{N} \text{ is odd, but } \sum_{n=1}^{N} X_n > -M). \]

Here, using the fact that $\prod_{n=1}^{\infty} \left(1 - \frac{1}{P_n}\right) = 0$, we obtain

\[ P(\tilde{N} \text{ is odd, but } \sum_{n=1}^{N} X_n > -M) \leq P(\tilde{N} \leq M) = \prod_{n=[M]+1}^{N} \left(1 - \frac{1}{P_n}\right) \rightarrow 0 \quad \text{as} \quad N \to \infty, \]

where, the notation $[z]$ denotes the largest integer which is no more than $z$. So that

\[ \cdot P(\sum_{n=1}^{N} X_n \leq -M) = P(\tilde{N} \text{ is odd}) + o(1) = P_N^{(1)} + o(1). \]
Similarly,

\[ P(\sum_{n=1}^{N} X_n \geq M) = P(\hat{N} \text{ is even}) + o(1) \Delta P_{N}^{(2)} + o(1). \]

Clearly,

\[ P_{N}^{(1)} + P_{N}^{(2)} = 1. \tag{4.8} \]

Furthermore, since \(1/p_m - 1/p_{m+1} \geq 1/p_m p_{m+1}, m \geq 0(p_0 \Delta 1),\) we have

\[ \frac{1}{p_m} \prod_{n=m+1}^{N} (1 - \frac{1}{p_n}) \geq \frac{1}{p_{m+1}} \prod_{n=m+2}^{N} (1 - \frac{1}{p_n}) \quad \text{for } 0 \leq m \leq N - 1 \text{(where } \prod_{n=N+1}^{N} (1 - \frac{1}{p_n}) \Delta 1),\]

combining this with\( (4.6) \) implies

\[ 0 \leq P_{N}^{(2)} - P_{N}^{(1)} = \sum_{m=0}^{N} (-1)^m \frac{1}{p_m} \prod_{n=m+1}^{N} (1 - \frac{1}{p_n}) \leq \prod_{n=1}^{N} (1 - \frac{1}{p_n}) \to 0 \quad \text{as } N \to \infty. \tag{4.9} \]

By \( (4.8), (4.9) \), clearly, \( \lim_{N \to \infty} P_{N}^{(1)} = \lim_{N \to \infty} P_{N}^{(2)} = \frac{1}{2}, \) so that

\[ \lim_{N \to \infty} P(\sum_{n=1}^{N} X_n \leq -M) = \lim_{N \to \infty} P(\sum_{n=1}^{N} X_n \geq M) = \frac{1}{2}. \]

Hence, for any real number \( x, \) and numbers \( x_1 < x_2, \) we have

\[ \lim_{N \to \infty} F_{N}(x) \equiv \frac{1}{2} \in (0, 1) \quad \text{and} \quad \lim_{N \to \infty} (F_{N}(x_2) - F_{N}(x_1)) = 0. \]

5 Proof of Theorem 3

Trivially, the weak convergence of \( F_N \) implies \((1.6), (1.7),\) then we only need to prove the \(''sufficient''\) assertion. The following proof is formulated in two steps as follows.

Step 1 First we prove the strict monotonicity of the distribution \( G \) in \((3.1).\) Since the set formed by all medians of a random variable is always a closed interval \([a, b]\) with \( a \leq b, \) we may select the maximum or minimum median \( \mu(X_n) \) of \( X_n \) such that

\[ \mu(X_n) \in R(X_n). \tag{5.1} \]

By \((1.8), (2.2)\) we can obtain

\[ \lim_{n \to \infty} \mu(X_n) = 0. \tag{5.2} \]

In fact, by \((1.8),\) there exists \( n_2 \in N \) such that

\[ P(X_n = 0) > q \equiv \frac{1}{2} \liminf_{n \to \infty} P(X_n = 0) > 0, \tag{5.3} \]

for all \( n > n_2, \) and by \((2.2),\)

\[ Z_n = X_n - \mu(X_n) \to 0, \text{ a.s. as } n \to \infty, \tag{5.4} \]

so that \((5.2)\) holds.
Let $\epsilon, 0 < \epsilon < 1$, be arbitrarily fixed. By (2.4), (5.2), (5.4), there exists $n_3 = n_3(\epsilon) > n_2$ such that
\[ |E^\frac{1}{2}Z_n^\frac{1}{2}| < \frac{\epsilon}{2}, |E(Z_n I(\{|Z_n| < 1\})| < \frac{\epsilon}{2} \text{ and } |b_n| < \epsilon, \] (5.5)
for all $n > n_3$. Moreover, for all $n > n_3$, we can assert
\[ \Delta_n^{(1)} = \Delta_n^{(1)}(\frac{\epsilon}{2}) \triangleq [\frac{-\epsilon}{2}, E^\frac{1}{2}Z_n^\frac{1}{2}] \cap R(Z_n) \neq \emptyset. \] (5.6)
In fact, by (2.6), (5.1), $0 \in R(Z_n)$, set $\alpha = \inf([-\epsilon/2, \epsilon/2] \cap R(Z_n)) \in R(Z_n)$, clearly $-\epsilon/2 \leq \alpha \leq 0$. If $\alpha = -\epsilon/2$, (5.6) naturally holds; if $-\epsilon/2 < \alpha \leq 0$, we have
\[ E^\frac{1}{2}Z_n^\frac{1}{2} = E(Z_n I(\alpha \leq Z_n < \frac{\epsilon}{2})) \geq \alpha P(\alpha \leq Z_n < \frac{\epsilon}{2}) \geq \alpha, \] (5.6) still holds. Similarly, for all $n > n_3$, we have
\[ \Delta_n^{(2)} = \Delta_n^{(2)}(\frac{\epsilon}{2}) \triangleq [\frac{\epsilon}{2}, E^\frac{1}{2}Z_n^\frac{1}{2}] \cap R(Z_n) \neq \emptyset. \] (5.7)
Now we select the numbers $C_n^{(i)}, i = 1, 2, n > n_3$ as follows: (1) Set $C_n^{(1)} = \theta_n - E(X_n I(\{|X_n| < 1\})$ (where $\theta_n \in \Delta_n^{(1)}$), $C_n^{(2)} = -b_n$ if $b_n < 0$; (2) Set $C_n^{(1)} = -b_n, C_n^{(2)} = \theta_n - E(X_n I(\{|X_n| < 1\})$ (where $\theta_n \in \Delta_n^{(2)}$), if $b_n > 0$; (3) Set $C_n^{(1)} = C_n^{(2)} = 0$ if $b_n = 0$. By (2.6), (5.3) and the construction of $\theta_n$, we see $C_n^{(1)} \in R(Y_n)$. By (5.6), (5.7), $|\theta_n| < \epsilon/2$, then by (5.5), we see $|C_n^{(1)}| < \epsilon$. Finally, by (2.2), (3.5) and the construction of $\theta_n$, we obtain
\[ \sum_{n=n_3+1}^{\infty} C_n^{(1)} = -\sum_{n=n_3+1}^{\infty} b_n + \sum_{n=n_3+1}^{\infty} \sum_{\{b_n > 0\}} (\theta_n - E^\frac{1}{2}Z_n^\frac{1}{2} - E(Z_n I(\frac{\epsilon}{2} \leq |Z_n| < 1))) = -\infty. \]
And similarly, \[ \sum_{n=n_3+1}^{\infty} C_n^{(2)} = \infty. \] Therefore by Lemma 3 we come to learn that $G$ is strictly increasing.

**Step 2** Next we prove the weak convergence of $F_N$. Now we may readily obtain the convergence of $A_N$. In fact, if $A_N$ diverges, there exist two subsequence $N_*$ and $N^*$ such that (4.3) holds. Set $s, t \in C(G)$, satisfy $x_0 - A^* < s < t < x_0 - A_*$, where the notation $C(G)$ denotes the set of all continuity points of $G$. By (1.7), (3.2) and the strict monotonicity of $G$, we have reached a contradiction by deducing
\[ L_0 = \liminf_{N \to \infty} F_N(x_0) \leq \liminf_{N^* \to \infty} F_{N^*}(x_0) = \liminf_{N^* \to \infty} G_{N^*}(x_0 - A_{N^*}) \]
\[ \leq \liminf_{N^* \to \infty} G_{N^*}(s) = G(s) < G(t) \leq \limsup_{N \to \infty} F_N(x_0) = L_0. \]
Finally by Lemma 4 we obtain the desired conclusion.

This completes the proof of Theorem 3.

**References**