A sharp inequality for the tail probabilities of sums of i.i.d. r.v.'s with dominatedly varying tails

TANG Qihe (唐启鹤)¹ & YAN Jia’an (颜嘉安)²

1. Department of Quantitative Economics, University of Amsterdam, Roetersstraat 11, 1018 WB Amsterdam, The Netherlands (email: tangqihe@263.net);
2. Academy of Mathematics and System Sciences, Chinese Academy of Sciences, Beijing 100080, China

Correspondence should be addressed to Yan Jia’an (email: jayan@mail.amt.ac.cn)

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Abstract Let $F$ be a distribution function supported on $(-\infty, \infty)$ with a finite mean $\mu$. In this note we show that if its tail $\bar{F} = 1 - F$ is dominatedly varying, then for any $\gamma > \max\{\mu, 0\}$, there exist $C(\gamma) > 0$ and $D(\gamma) > 0$ such that

$$C(\gamma)n\bar{F}(x) \leq \frac{\bar{F}(x)}{\bar{F}(x) - \bar{F}(\gamma x)} \leq D(\gamma)n\bar{F}(x),$$

for all $n \geq 1$ and all $x \geq \gamma n$. This inequality sharpens a classical inequality for the subexponential distribution case.

Keywords: dominatedly varying tails, subexponential distribution, tail probabilities.

Throughout this paper, for a given random variable (r.v.) $X$ with a finite mean $\mu$, we denote by $F(x) = P(X \leq x)$ the distribution function (d.f.) of $X$ and by $\bar{F} = 1 - F$ its tail. We always assume $F(x) < 1$ for any real value of $x$.

We say $X$ (or its d.f. $F$) is heavy-tailed on the right if

$$E\exp\{hX\} = \infty \text{ for any } h > 0.$$

See, for example, refs. [1–3] for the definitions of many types of heavy-tailed subclasses and their applications in context of insurance and finance. The most important heavy-tailed subclass is the subexponential class (denoted by $S$). By definition, a d.f. $F$ supported on $[0, \infty)$ is in $S$ if and only if

$$\lim_{x \to \infty} \frac{F^{n}(x)}{F(x)} = n$$

for all $n \geq 2$ (or equivalently for some $n \geq 2$), where $F^{n}$ denotes the $n$-fold convolution of $F$. Another well-known heavy-tailed subclass is called the dominated variation class (denoted by $D$). A d.f. $F$ supported on $(-\infty, \infty)$ is in $D$ if and only if its tail $\bar{F}$ is of dominated variation in the sense that

$$\limsup_{x \to \infty} \frac{F(\theta x)}{F(x)} < \infty$$

for any $0 < \theta < 1$ (or equivalently for some $0 < \theta < 1$). $D$ is a very large heavy-tailed subclass and is very useful in applied probability. It has a close relationship with $S$ (see ref. [1] for details).
In the fields of applied probability, a famous and classical probability inequality states that if $F$ is subexponential, then for any $\varepsilon > 0$ there exists an $A(\varepsilon) > 0$ such that

$$\overline{F^\gamma(x)} \leq A(\varepsilon)(1 + \varepsilon)^n F(x),$$

(0.2)

for all $n \geq 1$ and all $x \geq 0$. This inequality was first obtained in refs. [4, 5]. Recently, refs. [1—3] showed some important applications of (0.2) to the fields of applied probability and risk theory. The purpose of this note is to establish a similar, but much sharper, inequality for distributions in class $D$.

1 Main results

Theorem 1. Let $F \in \mathcal{D}$ with a finite mean $\mu$. Then

(i) for any $\gamma > 0$ there exists some constant $C(\gamma) > 0$ such that

$$\overline{F^\gamma(x)} \geq C(\gamma)n F(x)$$

(1.1)

for all $n \geq 1$ and all $x \geq \gamma n$;

(ii) for any $\gamma > \mu^+$ = $\max \{\mu, 0\}$ there exists some constant $D(\gamma) > 0$ such that

$$\overline{F^\gamma(x)} \leq D(\gamma)n F(x)$$

(1.2)

for all $n \geq 1$ and all $x \geq \gamma n$.

We remark that the $x$-region used in Theorem 1 is $T_n = [\gamma n, \infty]$, which is the most important case in applications and is crucial for research on precise large deviations; please refer to ref. [6] for more details. We also remark that Cline and Hsing$^1$ obtained the similar results as in (1.1) and (1.2) earlier in the 1990s for a more flexible $x$-region $T_n$ than ours. The conditions they assumed on the d.f. $F$, however, are too complicated and unnatural. For the most useful case $T_n = [\gamma n, \infty]$, their results will hold under the conditions $F \in \mathcal{D}$, $\mu = 0$ and $EX^2 < \infty$, where $X$ is some r.v. distributed by $F$.

Obviously, if $F \in \mathcal{D}$, then for any $c > 0$, as $x \to \infty$, $\overline{F(cx)}$ and $\overline{F(x)}$ are of the same order in the sense that

$$0 < \liminf_{x \to \infty} \frac{\overline{F(cx)}}{\overline{F(x)}} \leq \limsup_{x \to \infty} \frac{\overline{F(cx)}}{\overline{F(x)}} < \infty.$$

(1.3)

We designate this fact by $\overline{F(cx)} \simeq \overline{F(x)}$. We also have

$$\overline{F(x + M)} \simeq \overline{F(x)}$$

(1.4)

for any real number $M$.

From Theorem 1 we know that, for any $\gamma > \mu^+$, it holds that

$$0 < \liminf_{n \to \infty} \inf_{x \geq \gamma n} \frac{\overline{F^\gamma(x)}}{n F(x)} \leq \limsup_{n \to \infty} \sup_{x \geq \gamma n} \frac{\overline{F^\gamma(x)}}{n F(x)} < \infty.$$  

(1.5)

That is, $\overline{F^\gamma(x)} \simeq n F(x)$ holds uniformly for $x \geq \gamma n$ as $n \to \infty$. Furthermore, we have

Corollary 1. Let $F \in \mathcal{D}$ with a finite mean $\mu$. Then for any $\gamma > \mu$, there exist some positive constants $C(\gamma)$ and $D(\gamma)$ such that

$$C(\gamma)n F(x - n\mu) \leq \overline{F^\gamma(x)} \leq D(\gamma)n F(x - n\mu)$$

(1.6)

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$^1$ Cline, D. B. H., Hsing, T., Large deviation probabilities for sums and maxima of random variables with heavy or subexponential tails, Texas A&M University, 1991, preprint.
holds for all \( n \geq 1 \) and all \( x \geq \gamma n \).

**Proof.** Let \( \{X_n, n \geq 1\} \) be a sequence of i.i.d. r.v.’s with the d.f. \( F \). We have

\[
F^{*n}(x) = P\left( \sum_{i=1}^{n} (X_i - \mu) > x - n\mu \right).
\]

Note that \( E(X_i - \mu) = 0 \), and that, \( x - n\mu \geq \tilde{\gamma}n \) with \( \tilde{\gamma} = \gamma - \mu > 0 \) for \( x \geq \gamma n \). It follows from (1.5) that, uniformly for \( x \geq \gamma n \) as \( n \to \infty \),

\[
P\left( \sum_{i=1}^{n} (X_i - \mu) > x - n\mu \right) \simeq nP \left( X_1 > x - n\mu \right) \simeq nF(x - n\mu), \tag{1.7}
\]

from which (1.6) follows. Here, in the last step of (1.7) we have used the fact in (1.4). This ends the proof of Corollary 1.

## 2 Lemmas

Before giving the proof of Theorem 1 we prepare some preliminaries. Apart from (1.3) and (1.4), the following lemma shows another property of distributions in class \( D \).

**Lemma 1.** Let \( F \in D \). If \( G \) is a d.f. such that \( \limsup_{x \to \infty} G(x)/F(x) < \infty \), then \( F*G \in D \) and

\[
F*G(x) \simeq F(x) \quad \text{as} \quad x \to \infty. \tag{2.1}
\]

**Proof.** It suffices to prove (2.1), because (2.1) and the condition \( F \in D \) imply \( F*G \in D \). For any \( 0 < l < 1 \), we have

\[
F*G(x) = \left( \int_{-\infty}^{lx} + \int_{lx}^{\infty} \right) F(x-t)G(dt) \leq F((1-l)x) + G(lx) \simeq F(x). \tag{2.2}
\]

On the other hand, we can choose some constant \( M > 0 \) sufficiently large such that \( G(-M) > 0 \). For this \( M \) we have

\[
F*G(x) \geq \int_{-M}^{\infty} F(x-t)G(dt) \geq F(x+M)G(-M) \simeq F(x). \tag{2.3}
\]

Combining (2.2) with (2.3) implies (2.1). This ends the proof of Lemma 1.

By induction, we obtain the following consequence of Lemma 1, which is essential for the proof of Theorem 1.

**Lemma 2.** Let \( F \in D \). Then for any \( n \geq 1 \), we have \( F^{*n} \in D \) and that

\[
F^{*n}(x) \simeq F(x) \quad \text{as} \quad x \to \infty. \tag{2.4}
\]

## 3 Proof of Theorem 1

First of all, we introduce a sequence of i.i.d. r.v.’s \( \{X_n, n \geq 1\} \) with the d.f. \( F \). Denote its \( n \)th partial sum by \( S_n \).

The lower bound in (1.1) for the case where \( \mu = 0 \) was obtained early in Cline and Hsing\(^1\). For the sake of self-containment we copy here their proof with a slight adjustment in the last process.
step. Let $\lambda > 0$. We have
\[
F_n(x) \geq P \left( S_n > x, \max_{1 \leq i \leq n} X_i > \lambda x \right)
\geq \sum_{i=1}^{n} P \left( S_n > x, X_i > \lambda x \right) - \sum_{1 \leq i < j \leq n} P \left( S_n > x, X_i > \lambda x, X_j > \lambda x \right)
\geq \sum_{i=1}^{n} P \left( S_n - X_i > (1 - \lambda) x, X_i > \lambda x \right) - (nF(\lambda x))^2
= nF(\lambda x) \left( P \left( S_{n-1} > (1 - \lambda) x \right) - nF(\lambda x) \right).
\]
(3.1)
Choosing $\lambda > \max \{(1 - \mu/\gamma), 1\}$ in (3.1), one can easily check that as $n \to \infty$,
\[
P \left( S_{n-1} > (1 - \lambda) x \right) \to 1 \quad \text{and} \quad nF(\lambda x) \to 0
\]
uniformly for $x \geq \gamma n$. So there exists some constant $D_1(\gamma) > 0$ such that $\overline{F}_n(x)$ is no less than $D_1(\gamma)nF(\lambda x)$. This, together with the definition of (0.1), implies the desired lower bound in (1.1).

Now we turn to prove the upper bound in (1.2). We write
\[
X_n = X_n I(X_n \leq \theta x) \quad \text{and} \quad \overline{S}_n = \sum_{i=1}^{n} \overline{X}_i \quad \text{for} \quad n \geq 1,
\]
where $0 < \theta < 1$ is arbitrarily fixed. Obviously,
\[
\overline{F}_n(x) \leq P \left( \max_{1 \leq i \leq n} X_i > \theta x \right) + P \left( \max_{1 \leq i \leq n} X_i \leq \theta x, S_n > x \right)
\leq nF(\theta x) + P \left( \overline{S}_n > x \right).
\]
(3.2)
By the definition of (0.1), there exists $C_1 > 0$ depending on $\theta$ such that
\[
nF(\theta x) \leq C_1 nF(x), \quad \text{for all} \quad n \geq 1, \quad x \geq \gamma n.
\]
(3.3)
This gives the upper bound for the first term in (3.2). Now we estimate the second term in (3.2).

We put
\[
m_+ = E[X I(X \geq 0)], \quad m_- = E[X I(X < 0)] \quad \text{and} \quad a = \max \{-\log(nF(x)), 1\}.
\]
Clearly, $a$ tends to $\infty$ uniformly for $x \geq \gamma n$ as $n \to \infty$. For any $h > 0$, we have
\[
\frac{P \left( \overline{S}_n > x \right)}{nF(x)} \leq \frac{e^{-h x} E e^{h \overline{S}_n}}{nF(x)}
\leq e^{-h x + a} \left\{ \left( \int_{-\infty}^{\theta x} (e^{h t} - 1)F(dt) + 1 \right)^n \right\}
\leq \exp \left\{ n \left( \int_{-\infty}^{0} + \int_{0}^{\theta x/a} + \int_{\theta x/a}^{\theta x} \right) (e^{h t} - 1)F(dt) - hx + a \right\}
= \exp \left\{ n \{ I_1 + I_2 + I_3 \} - hx + a \right\}.
\]
(3.4)
Next we deal with $I_1, I_2$ and $I_3$ respectively. Since for $t \leq 0$,\[
0 \leq \frac{e^{h t} - 1 - ht}{h} \leq |t|.
\]
by the dominated convergence theorem we have
\[
\lim_{h \to 0} \frac{\int_{-\infty}^{\theta x} (e^{h t} - 1)F(dt)}{h} = \int_{-\infty}^{0} \lim_{h \to 0} \frac{e^{h t} - 1 - ht}{h} F(dt) + m_- = m_-
\]
Thus, if \( m_- \neq 0 \), there exists a function \( \delta \) with \( \delta(h) \) tending to 0 as \( h \searrow 0 \) such that
\[
I_1 = \int_{-\infty}^{0} (e^{h} - 1) F(dt) = (1 + \delta(h)) hm_-.
\] (3.5)

If \( m_- = 0 \), then \( I_1 \leq 0 \) and (3.5) holds with \( \delta(h) = 0 \) and the last symbol \( \ll \) being replaced by \( \ll \). As for \( I_2 \), by an elementary inequality that \( e^u - 1 \leq u e^u \) for any real number \( u \),
\[
I_2 = \int_{0}^{\theta x/a} (e^{h} - 1) F(dt) \leq \theta e^{h} \int_{0}^{\theta x/a} t F(dt) \leq \theta \int_{0}^{\infty} t F(dt) \leq \theta hm_+ e^{h} x/a.
\] (3.6)

Now we turn to treat \( I_3 \) in (3.4). An existing result in Proposition 2.2.1 of ref. [7] states that, for \( F \in D \), there exist positive \( x_0, \rho \) and \( B \) such that
\[
\mathcal{F}(\theta x)/\mathcal{F}(x) \leq B\theta^{-\rho}
\] (3.7)
uniformly for all \( x \geq x_0/\theta \) and all \( 0 < \theta < 1 \). We remark that (3.7) holds even for the case where \( \theta = \theta(x) \) tends to 0 as \( x \to \infty \). Set
\[
h = \frac{a - K \rho \log a}{K\theta x}
\]
in (3.4), where \( \rho \) is from (3.7) and \( K > 1 \) will be determined later. Clearly, \( h \) tends to 0 uniformly for all \( x \geq \gamma n \) as \( n \to \infty \). Since \( a \) tends to \( \infty \) uniformly for \( x \geq \gamma n \), there exists an integer \( N_1 \) such that \( h > 0 \) and \( a > 1 \) for all \( x \geq \gamma N_1 \). By (3.7) with \( \theta \) begin replaced by \( \theta/a \), we obtain the following upper bound for \( I_3 \):
\[
I_3 = \int_{0}^{\theta x/a} (e^{h} - 1) F(dt) \leq e^{h} \mathcal{F}(\theta x/a)
\]
\[
\leq \exp \left\{ \frac{a - K \rho \log a}{K} B \left( \theta/a \right)^{-\rho} \mathcal{F}(x) = B\theta^{-\rho} (n\mathcal{F}(x))^{-1/K} \mathcal{F}(x). \right\}
\] (3.8)

Substituting (3.5), (3.6) and (3.8) into (3.4) and using the fact that \( h\theta x/a \leq 1/K \), we obtain the following upper bound for the right-hand side of (3.4):
\[
\exp \left\{ \frac{1}{1 + \delta(h)} nhm_- + nhm_+ e^{1/K} + B\theta^{-\rho} (n\mathcal{F}(x))^{-1-1/K} - hx + a \right\}. \] (3.9)

Noting that \( n\mathcal{F}(x) \to 0 \) uniformly for \( x \geq \gamma n \) as \( n \to \infty \), we obtain that, for some sufficiently large \( C_2 > 0 \), the right-hand side of (3.9) can further be bounded by
\[
C_2 \exp \left\{ \frac{1}{1 + \delta(h)} nhm_- + nhm_+ e^{1/K} - hx + a \right\}
\]
\[
= C_2 \exp \left\{ \frac{a - K \rho \log a}{K} \left( \frac{1 + \delta(h)}{x} n + \left( \frac{e^{1/K} - 1}{\gamma} \delta(h) \right) n + a \right) \right\}
\]
\[
\leq C_2 \exp \left\{ \frac{a - K \rho \log a}{K} \left( \frac{1 + \delta(h)}{x} n + \left( \frac{e^{1/K} - 1}{\gamma} \delta(h) \right) n + a \right) \right\}. \] (3.10)

Since \( \gamma > \mu^+ \), we can choose \( K > 0 \) sufficiently large such that
\[
\frac{\mu^+}{\gamma} + \left( \frac{e^{1/K} - 1}{\gamma} \right) m_+ < 1,
\]
then we choose \( \theta = \theta(K) > 0 \) sufficiently small such that
\[
\frac{1}{K\theta} \left( \frac{\mu^+}{\gamma} + \left( \frac{e^{1/K} - 1}{\gamma} \right) m_+ \right) + 1 < 0. \] (3.11)

It follows from (3.10) and (3.11) that
\[
\lim_{n \to \infty} \sup_{x \geq \gamma n} \frac{P(X_n > x)}{n\mathcal{F}(x)} = 0.
\]
Consequently, there exists some integer \( N = N(K, \theta) > N_1 \) such that
\[
\sup_{n \geq N, \; x \geq \gamma n} \frac{P\left( S_n > x \right)}{nF(x)} < \infty.
\]
This, together with (3.3), implies that
\[
\sup_{n \geq N, \; x \geq \gamma n} \frac{F^n(x)}{nF(x)} < \infty.
\]
As for the case where \( 1 \leq n \leq N \) and \( x \geq \gamma n \), from Lemma 2 we have
\[
\sup_{1 \leq n \leq N, \; x \geq \gamma n} \frac{F^n(x)}{nF(x)} \leq \sum_{n=1}^{N} \sup_{x \geq \gamma n} \frac{F^n(x)}{nF(x)} < \infty.
\]
Thus, there exists a constant \( D(\gamma) > 0 \) such that the upper bound in (1.2) holds. This ends the proof of Theorem 1.

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