Abstract

In this paper, we propose a customer-arrival-based insurance risk model, in which customers' potential claims are described as independent and identically distributed heavy-tailed random variables and premiums are the same for each policy. We obtain some precise large deviation results for the prospective-loss process under a mild assumption on the random index (in our case, the customer-arrival process), which is much weaker than that in the literature.

Keywords: Insurance risk model; point process; precise large deviation; subexponentiality

AMS 2000 Subject Classification: Primary 60F10; 60F05; 60G50
Secondary 60K10; 62P05

1. Introduction

In the classical insurance risk model (see Embrechts et al. (1997) and Rolski et al. (1999)), the surplus is described as the initial surplus plus the premium income with the claims taken off. A compound process is often used to model this surplus process, in which the claim-number process is a counting process, and the claim sizes are assumed to form a sequence of independent and identically distributed (i.i.d.) random variables. In the classical model, the premium rate is assumed to be a constant. Recently, there have been some works which extend the classical model by assuming that the premium is also random. However, by doing so, they encounter the problem of dependence between the premium process and the claim-arrival process, and the problem becomes difficult.

In this paper we propose a different way of modelling the surplus of an insurance company. Rather than counting the number of claims, we count the number of customers. The model can be described as follows:

• The individual customer-arrival process is \( N(t) = \max\{k \geq 0 : \sigma_k \leq t\} \), where \( \sigma_k \) is the time at which the \( k \)th customer arrives, \( \sigma_0 = 0 \). Naturally, we assume that \( N(t) \) is a nonnegative and integer-valued stochastic process with a mean function \( \lambda(t) < \infty \) for any \( t > 0 \) but \( \lambda(t) \to \infty \) as \( t \to \infty \). In our setup, \( N(t) \) can be a Poisson process, Cox...
process or a marked point process. We do not exclude the possibility of more than one customer arriving at the same time.

- At the time $\sigma_k$, the $k$th customer buys an insurance contract, $k \geq 1$. The insurance company will therefore bear a risk from this policy holder within a fixed term, say $\tau$. The term $\tau$ can be one month, one year or infinite.

- Assume that the total potential claims due to the $k$th customer within the term $\tau$ is $X_k$, and that $\{X_k, k \geq 1\}$ forms a sequence of i.i.d. nonnegative random variables with a common distribution function $F$ and a finite mean $\mu$. This assumption means that we are considering all the customers coming from a certain category of the population. All the individuals in this category are indistinguishable in terms of the total potential claims. The price of each policy is $(1 + \delta)\mu$, where the positive constant $\delta$ can be interpreted as the safety loading coefficient. Clearly, the insurance company’s total net risk coming from the $k$th policy holder is $X_k - (1 + \delta)\mu$.

- We assume that the interest rate is zero. Therefore, the surplus process of the company within the period $[0, t]$ can be written as $U(t) = x - W(t)$, where $x$ denotes the initial reserve of the company and $W(t)$ denotes the prospective-loss process

$$W(t) = \sum_{k=1}^{N(t)} (X_k - (1 + \delta)\mu), \quad t \geq 0, \quad (1.1)$$

which is, of course, a very important object in risk management.

In this paper, we address the large deviation problems for the prospective-loss process $W(t)$. For some fundamental results on precise large deviations and their applications in insurance and finance, we refer the reader to Cline and Hsing (1991), Asmussen and Klüppelberg (1996), Klüppelberg and Mikosch (1997), Embrechts et al. (1997, Sections 8.6 and 8.7), Mikosch and Nagaev (1998) and Tang et al. (2001), among others.

This paper is organized as follows. Section 2 presents some mathematical concepts and notation, recalls some key results on precise large deviations and states our main results. The proofs of the main results are presented in Section 3.

### 2. Notation and main results

Given a nonnegative random variable $X$ with a finite mean $\mu$, its distribution function is denoted by $F(x) = P(X \leq x)$ and its tail by $\overline{F} = 1 - F$. We say that $X$ (or its distribution function $F$) is heavy tailed if it has no exponential moments. See, for example, Embrechts et al. (1997) for the definitions of many types of heavy-tailed subclasses and their applications to insurance and finance. The most well-known subclass of heavy-tailed distributions is the subexponential class, denoted by $\delta$. By definition, a distribution function $F$ with support $[0, \infty)$ belongs to $\delta$ if, for some $n \geq 2$ (or, equivalently, for any $n \geq 2$),

$$\lim_{x \to \infty} \frac{F_{n,x}(x)}{F(x)} = n. \quad (2.1)$$

Recall that (see for example Lemma 1.3.5 in Embrechts et al. (1997)), for $F \in \delta$,

$$\lim_{x \to \infty} \frac{\overline{F}(x + L)}{\overline{F}(x)} = 1 \quad \text{for any fixed } L > 0. \quad (2.2)$$
Another well-known heavy-tailed subclass is the class $\mathcal{R}$, which consists of distribution functions with regularly varying tails. A natural extended version of $\mathcal{R}$ is the so-called class $\text{ERV}$ (extended regularly varying class). By definition, a distribution function $F$ with support $[0, \infty)$ belongs to $\text{ERV}$ if

$$y^{-\beta} \leq \liminf_{x \to \infty} \frac{F(xy)}{F(x)} \leq \limsup_{x \to \infty} \frac{F(xy)}{F(x)} \leq y^{-\alpha}$$

for any $y > 1$ (2.3)

for some $\alpha$ and $\beta$ with $1 < \alpha \leq \beta < \infty$ (write $F \in \text{ERV}(-\alpha, -\beta)$ for simplicity). See Bingham et al. (1987, pp. 61–76), Cline and Hsing (1991) and references therein for the introduction and deeper investigations of the class $\text{ERV}$.

Henceforth, $\{X_k, k \geq 1\}$ denotes a sequence of i.i.d., nonnegative random variables with a generic random variable $X$ and a generic distribution function $F$ belonging to $\text{ERV}(-\alpha, -\beta)$, and $\{N(t), t \geq 0\}$ denotes a nonnegative, integer-valued process, which is independent of the sequence $\{X_k, k \geq 1\}$. We assume that $\lambda(t) = \mathbb{E} N(t) < \infty$ for all $t \geq 0$ but $\lambda(t) \to \infty$. All limit relations, unless otherwise stated, are for $t \to \infty$ (or, consequently, for $\lambda(t) \to \infty$).

We denote the partial sum of $\{X_k, k \geq 1\}$ by $S_n = \sum_{k=1}^n X_k$, as usual.

The following result, the so-called precise large deviation of $S_n$, is an easy consequence of a very general result obtained in Cline and Hsing (1991) (see also Klüppelberg and Mikosch (1997)): for any fixed $\gamma > 0$,

$$\mathbb{P}\left(\sum_{k=1}^n (X_k - \mu) > x\right) \sim n F(x) \quad \text{uniformly for } x \geq \gamma n \text{ as } n \to \infty. \quad (2.4)$$

Recently, Klüppelberg and Mikosch (1997) extended (2.4) to the case of random sums. They showed that

$$\mathbb{P}\left(\sum_{k=1}^{N(t)} X_k - \mathbb{E} \sum_{k=1}^{N(t)} X_k > x\right) \sim \lambda(t) F(x) \quad \text{uniformly for } x \geq \gamma \lambda(t), \quad (2.5)$$

under the following two conditions.

**Assumption A.** We have

$$\frac{N(t)}{\lambda(t)} \xrightarrow{p} 1.$$

**Assumption B.** For any constant $\theta > 0$ and some small $\varepsilon > 0$,

$$\sum_{n>(1+\theta)\lambda(t)} \mathbb{P}(N(t) = n)(1 + \varepsilon)^n = o(1).$$

Clearly, Assumption B is quite a strong condition. Even the commonly used renewal counting process may fail to satisfy this condition. Recently, Tang et al. (2001, Theorem 2.1) weakened Assumption B to the following condition for some given $\beta > 1$.

**Assumption C$_\beta$.** For any fixed small constant $\theta > 0$ and some small $\varepsilon > 0$,

$$\mathbb{E} N(t)^{\beta + \varepsilon} 1_{(N(t)>(1+\theta)\lambda(t))} = O(\lambda(t)).$$
Tang et al. (2001) also proved that Assumption $C_\beta$ can be applied to at least the so-called compound renewal process.

The main purpose of the present paper is to investigate the precise large deviations of the prospective-loss process $W(t)$ of our insurance risk model. First of all, we state a precise large deviation result for the case of nonrandom sum as follows.

**Theorem 2.1.** Let $\{X_k, k \geq 1\}$ be a sequence of i.i.d. ERV$(-\alpha, -\beta)$ random variables with $1 < \alpha \leq \beta < \infty$. Then, for every fixed $\gamma > 0$, uniformly for $x \geq \gamma n$,

$$
P\left( \sum_{k=1}^{n}(X_k - (1+\delta)\mu) > x \right) \sim n\overline{F}(x + \delta n\mu), \quad n \to \infty, \quad (2.6)$$

and, uniformly for $x \gg n$,

$$
P\left( \sum_{k=1}^{n}(X_k - (1+\delta)\mu) > x \right) \sim n\overline{F}(x), \quad n \to \infty. \quad (2.7)$$

Here and henceforth, we write $x \gg n$ to mean that the related asymptotic result holds uniformly for $x \geq f(n)n$ for any choice of sequence $f(n)$ such that $0 \leq f(n) \to \infty$ as $n \to \infty$. The notation $x \gg \lambda(t)$ can be understood in a similar way.

**Proof.** The result (2.6) is a direct consequence of (2.4). Furthermore, from the definition of (2.3), we can obtain that

$$
\overline{F}(x + o(x)) \sim \overline{F}(x), \quad x \to \infty, \quad (2.8)
$$

for any $o(x)$, which implies that $\overline{F}(x + \delta n\mu) \sim \overline{F}(x)$ holds uniformly for $x \gg n$ as $n \to \infty$. This proves the assertion (2.7).

We now proceed to state the main results of the paper. The following result extends the precise large deviation results (2.6) and (2.7) to the random sums $W(t)$ in (1.1) under Assumption A on $N(t)$.

**Theorem 2.2.** Let $\{X_k, k \geq 1\}$ be a sequence of i.i.d. ERV$(-\alpha, -\beta)$ random variables with $1 < \alpha \leq \beta < \infty$. We assume that $\{X_k, k \geq 1\}$ is independent of the nonnegative, integer-valued process $\{N(t), t \geq 0\}$. Furthermore, we suppose that $N(t)$ satisfies Assumption A. Then, for any fixed $\gamma > 0$,

$$
P\left( \sum_{k=1}^{N(t)}(X_k - (1+\delta)\mu) > x \right) \sim \lambda(t)\overline{F}(x + \delta \lambda(t)\mu) \quad (2.9)
$$

uniformly for $x \geq \gamma \lambda(t)$.

By (2.8), we obtain a corollary of Theorem 2.2.

**Corollary 2.1.** Under the conditions of Theorem 2.2,

$$
P\left( \sum_{k=1}^{N(t)}(X_k - (1+\delta)\mu) > x \right) \sim \lambda(t)\overline{F}(x) \quad (2.10)
$$

uniformly for $x \gg \lambda(t)$.
Note that the way of centring the collective risks in (2.9) and (2.10) differs from that in (2.5). This naturally raises a question: how big is the difference between the asymptotics (2.5) and (2.9) (or (2.10))? Our answer is: under Assumption C, they are equivalent to each other. In fact, we obtain a slightly more general result.

**Theorem 2.3.** Let \( \{X_k, k \geq 1\} \) be a sequence of i.i.d. ERV\((-\alpha, -\beta)\) random variables with \(1 < \alpha \leq \beta < \infty\). We assume that \( \{X_k, k \geq 1\} \) is independent of the nonnegative, integer-valued process \( \{N(t), t \geq 0\} \). Then, under Assumption C, for any fixed \( \gamma > 0 \),

\[
P\left( \sum_{k=1}^{N(t)} X_k - E \sum_{k=1}^{N(t)} X_k > x \right) \sim \lambda(t) \mathcal{F}(x) \quad (2.11)
\]

uniformly for \( x \geq \gamma \lambda(t) \). If we further restrict the \( x \)-region to \( x \gg \lambda(t) \), then, uniformly,

\[
P\left( \sum_{k=1}^{N(t)} X_k - E \sum_{k=1}^{N(t)} X_k > x \right) \sim P\left( \sum_{k=1}^{N(t)} (X_k - (1 + \delta)\mu) > x \right) \sim \lambda(t) \mathcal{F}(x). \quad (2.12)
\]

Solvency is one of main concerns of insurance companies. We want to be sure that prospective loss at any moment \( t \) within the interval \([0, T]\) does not exceed \( x \). Therefore, we want to find the ‘solvency’ probability (also known as the ruin probability in a finite time horizon) \( P(\sup_{0 \leq t \leq T} W(t) > x) \). The following theorem gives an asymptotic relation between the ‘solvency’ probability and the tail probability of the potential claim of each customer. We will require the following assumption.

**Assumption D.** For a given \( T > 0 \) and for some \( \varepsilon > 0 \),

\[
E(1 + \varepsilon)^{N(T)} < \infty.
\]

**Theorem 2.4.** Let \( \{X_k, k \geq 1\} \) be a sequence of i.i.d. subexponential random variables, which is independent of the nonnegative, integer-valued process \( \{N(t), t \geq 0\} \) and let Assumption D hold. Then

\[
P\left( \sup_{0 \leq t \leq T} \sum_{k=1}^{N(t)} (X_k - (1 + \delta)\mu) > x \right) \sim \lambda(T) \mathcal{F}(x), \quad x \to \infty. \quad (2.13)
\]

It is easy to see that Assumption B implies Assumption D, and the homogeneous Poisson arrival process satisfies Assumption D. We also remark that the asymptotics (2.13) is a rough result, which is insensitive to the safety loading coefficient \( \delta > 0 \).

### 3. Proofs of main results

#### 3.1. Preliminaries

It is well known that, when \( F \) is subexponential, the tail of its \( n \)-fold convolution is bounded by the tail of \( F \) in the following way: for any \( \varepsilon > 0 \), there exists an \( A(\varepsilon) > 0 \) such that, uniformly for all \( n \geq 1 \) and all \( x \geq 0 \),

\[
F^{\ast n}(x) \leq A(\varepsilon)(1 + \varepsilon)^{n} \mathcal{F}(x). \quad (3.1)
\]

See Embrechts et al. (1997, pp. 41–42), Rolski et al. (1999, p. 53), Asmussen (2000, p. 255) and references therein for the details of this classical inequality. Note that, in Klüppelberg and
Mikosch (1997), Assumption B was employed in order to use (3.1) in their proof. Tang et al. (2001, Lemma 3.2) found that (3.1) can be improved to
\[
F^{n\ast}(x) \leq Cn^{\beta + \epsilon}F(x)
\] (3.2)
uniformly for all \(n \geq 1\) and all \(x \geq 0\) if \(F \in \text{ERV}(-\alpha, -\beta)\), where \(C\) is a positive constant, independent of \(n\) and \(x\). Recently, Tang and Yan (2002) studied the two-sided bounds for the tail of the \(n\)-fold convolution of a distribution function \(F\) with a dominatedly varying tail. As a special case when \(F \in \text{ERV}(-\alpha, -\beta)\) with \(1 < \alpha \leq \beta < \infty\), the main result in Tang and Yan (2002) shows a sharper upper bound
\[
F^{n\ast}(x) \leq C(\gamma)nF(x)
\] (3.3)
uniformly for \(x > \gamma n\) and \(n \geq 1\), where \(\gamma > \mu\) is arbitrarily fixed and \(C(\gamma)\) is a positive constant, independent of \(n\) and \(x\). The inequality (3.3) will be our main tool in establishing Theorem 2.2 under Assumption A.

In the proof of Theorem 2.2, we will need the following result.

**Lemma 3.1.** Let \(\zeta(t)\) be a stochastic process with a mean function \(E \zeta(t) \to 1\). Then
\[
\zeta(t) \xrightarrow{p} 1
\] (3.4)
if and only if
\[
E \zeta(t) 1_{(|\zeta(t)| - 1 \geq \theta)} = o(1) \quad \text{for any } \theta > 0
\] (3.5)
or, equivalently,
\[
E \zeta(t) 1_{(|\zeta(t)| - 1 \geq \theta)} = o(1) \quad \text{for any } \theta > 0.
\] (3.6)

**Proof.** We will show that (i) (3.4) implies (3.6), (ii) (3.6) implies (3.5) and (iii) (3.5) implies (3.4).

(i) By the well-known dominated convergence theorem, we obtain, from (3.4), that, for any \(\theta > 0\),
\[
\lim_{t \to \infty} E \zeta(t) 1_{(|\zeta(t)| - 1 \geq \theta)} = 1.
\]
Therefore,
\[
E \zeta(t) 1_{(|\zeta(t)| - 1 \geq \theta)} = 1 - E \zeta(t) 1_{(|\zeta(t)| - 1 < \theta)} = o(1).
\]
(ii) That (3.6) implies (3.5) is trivial.
(iii) See Lemma 3.3 of Tang et al. (2001).

By this lemma, we can easily see that not only Assumption B but also Assumption \(C_\beta\) implies Assumption A.

### 3.2. Proof of Theorem 2.2

We divide the left-hand side of (2.9) into two summands:
\[
P\left(\sum_{k=1}^{N(t)} (X_k - (1 + \delta)\mu) > x\right) = I_1 + I_2,
\] (3.7)
where

\[ I_1 := \sum_{|n-\lambda(t)| \geq \theta \lambda(t)} P(S_n - (1 + \delta) \mu n > x) P(N(t) = n), \]

\[ I_2 := \sum_{|n-\lambda(t)| < \theta \lambda(t)} P(S_n - (1 + \delta) \mu n > x) P(N(t) = n) \]

for some arbitrarily chosen \( \theta \) with \( 0 < \theta < 1 \). First we deal with \( I_1 \). By (3.3),

\[ P(S_n - (1 + \delta) \mu n > x) \leq C n F(x + \delta \mu n) \leq C n F(x). \]

Thus,

\[ I_1 \leq C F(x) \mathbb{E} N(t) 1_{(N(t) - \lambda(t)| \geq \theta \lambda(t))} = o(\lambda(t) F(x)) = o(\lambda(t) F(x + \delta \lambda(t) \mu)), \quad (3.8) \]

where, in the penultimate step, we have used Lemma 3.1 with \( \zeta(t) = N(t)/\lambda(t) \), and in the last step we have used the definition (2.3) and the \( x \)-region given in the theorem. Now we deal with \( I_2 \). Note that, in this case, we have \( x \geq \gamma \lambda(t) \geq (\gamma/(1 + \theta)) n \). Thus, it follows from Theorem 2.1 that

\[ I_2 \sim \sum_{|n-\lambda(t)| < \theta \lambda(t)} n F(x + \delta \mu n) P(N(t) = n). \quad (3.9) \]

Thus, by Assumption A and Lemma 3.1 with \( \zeta(t) = N(t)/\lambda(t) \), we obtain that

\[ \frac{I_2}{F(x + \delta \lambda(t) \mu) \lambda(t)} \sim \sum_{|n-\lambda(t)| < \theta \lambda(t)} \frac{F(x + \delta \mu n)}{F(x + \delta \lambda(t) \mu) \lambda(t)} \mathbb{E} N(t) 1_{(N(t) - \lambda(t)| \leq \theta \lambda(t))} P(N(t) = n) \]

\[ \leq \sup_{x \geq \gamma \lambda(t)} \frac{F(x + \delta \mu (1 - \theta) \lambda(t))}{F(x + \delta \lambda(t) \mu)} \mathbb{E} N(t) \frac{\lambda(t)}{\lambda(t)} 1_{(N(t) - \lambda(t)| \leq \theta \lambda(t))} \]

\[ \sim \sup_{x \geq \gamma \lambda(t)} \frac{F(x + \delta \mu (1 - \theta) \lambda(t))}{F(x + \delta \lambda(t) \mu)}. \quad (3.10) \]

By the same approach, we also obtain the corresponding lower bound

\[ \inf_{x \geq \gamma \lambda(t)} \frac{F(x + \delta \mu (1 + \theta) \lambda(t))}{F(x + \delta \lambda(t) \mu)}. \quad (3.11) \]

We will use another equivalent description of ERV: \( F \in \text{ERV}(-\alpha, -\beta) \) for some \( \alpha \) and \( \beta \) with \( 1 < \alpha \leq \beta < \infty \) if and only if

\[ y^{-\alpha} \leq \liminf_{x \to \infty} \frac{F(xy)}{F(x)} \leq \limsup_{x \to \infty} \frac{F(xy)}{F(x)} \leq y^{-\beta} \text{ for any } y \text{ with } 0 < y < 1. \quad (3.12) \]
Using (3.12), and substituting (3.8), (3.9) and (3.10) into (3.7), we obtain that
\[
\limsup_{t \to \infty} \sup_{x \geq \gamma \lambda(t)} \frac{P(\sum_{k=1}^{N(t)} (X_k - (1 + \delta)\mu) > x)}{F(x + \delta \lambda(t)\mu)} \leq \limsup_{t \to \infty} \sup_{x \geq \gamma \lambda(t)} \frac{F((1 - \theta)(x + \delta \lambda(t)\mu))}{F(x + \delta \lambda(t)\mu)} \leq (1 - \theta)^{- \beta} \rightarrow 1 \quad \text{as } \theta \to 0.
\] (3.13)

Symmetrically, by (3.11),
\[
\liminf_{t \to \infty} \inf_{x \geq \gamma \lambda(t)} \frac{P(\sum_{k=1}^{N(t)} (X_k - (1 + \delta)\mu) > x)}{F(x + \delta \lambda(t)\mu)} \geq (1 + \theta)^{- \alpha} \rightarrow 1 \quad \text{as } \theta \to 0.
\] (3.14)

Thus, from (3.13) and (3.14), we finally obtain the desired result (2.9). This ends the proof of Theorem 2.2.

### 3.3. Proof of Theorem 2.3

For the \(x\)-region \(x \gg \lambda(t)\), from Corollary 2.1 above and Theorem 2.1 in Tang et al. (2001), we see that the asymptotics in (2.12) is valid. In the following, we aim to prove the asymptotics in (2.11) for the \(x\)-region \(x \geq \gamma \lambda(t)\) for any \(\gamma > 0\). Clearly, we only need to prove that
\[
P\left(\sum_{k=1}^{N(t)} (X_k - \mu) > x\right) \sim \lambda(t)F(x)
\] (3.15)
holds uniformly for \(x \geq \gamma \lambda(t)\) and for any fixed \(\gamma > 0\).

As noted at the end of Section 3.1, Assumption C\(\beta\) implies Assumption A. So, it is easy to see that there exists some \(\varepsilon(t)\) satisfying \(0 < \varepsilon(t) \to 0\) such that
\[
P\left(\frac{N(t)}{\lambda(t)} - 1 \bigg| < \varepsilon(t)\right) \to 1
\] (3.16)
(see Klüppelberg and Mikosch (1997)). Now, for a given \(\varepsilon > 0\), we choose some \(l(t)\) such that \(0 < l(t) \to \infty\) but \((1 + \varepsilon)l(t) = o(\lambda(t))\). We write \(P\sum_{k=1}^{N(t)} (X_k - \mu) > x\) as the sum \(\sum_{n=1}^{\infty} P(S_n - \mu n > x) P(N(t) = n)\) and divide this sum into five parts:
\[
\sum_{n=1}^{\infty} = \sum_{n=1}^{l(t)} + \sum_{n=l(t)+1}^{\lfloor (1-\varepsilon l(t))\lambda(t)\rfloor} + \sum_{n=\lfloor (1-\varepsilon l(t))\lambda(t)\rfloor+1}^{\lfloor (1+\varepsilon l(t))\lambda(t)\rfloor} + \sum_{n=\lfloor (1+\varepsilon l(t))\lambda(t)\rfloor+1}^{\infty} =: J_1 + J_2 + J_3 + J_4 + J_5.
\] (3.17)

Clearly, by the classical inequality (3.1),
\[
J_1 \leq A(\varepsilon)\overline{F}(x) \sum_{n=1}^{l(t)} (1 + \varepsilon)^n P(N(t) = n) \leq A(\varepsilon)\overline{F}(x)(1 + \varepsilon)^{l(t)} = o(\overline{F}(x)\lambda(t)).
\] (3.18)
Moreover, observe that the index \( n \) in \( J_2 \) and \( J_4 \) is such that \( x \geq \gamma \lambda(t) \geq (\gamma/(1 + \theta))n \), we have, from (2.4) and (3.16), that

\[
J_2 + J_4 \sim \left( \sum_{n=[\lambda(t)]+1}^{[1-\varepsilon(t)]\lambda(t)} nF(x)P(N(t) = n) + \sum_{n=[(1+\varepsilon(t))\lambda(t)]+1}^{[1+\varepsilon(t)]\lambda(t)} nF(x)P(N(t) = n) \right)
\]

\[
\leq (1 + \theta)\bar{F}(x)\lambda(t)P\left( \left\lfloor \frac{N(t)}{\lambda(t)} \right\rfloor + 1 \geq \varepsilon(t) \right) = o(\bar{F}(x)\lambda(t)).
\]  

(3.19)

As for \( J_3 \), by similar reasoning as above, we have that

\[
J_3 \sim \sum_{n=[(1-\varepsilon(t))\lambda(t)]+1}^{[1+\varepsilon(t)]\lambda(t)} n\bar{F}(x)P(N(t) = n)
\]

\[
\sim \lambda(t)\bar{F}(x)P([1-\varepsilon(t)]\lambda(t) + 1 \leq N(t) \leq [1+\varepsilon(t)]\lambda(t))
\]

\[
\sim \bar{F}(x)\lambda(t).
\]  

(3.20)

Finally, we deal with the last term \( J_5 \). By Assumption \( C_\beta \) and the inequality (3.2) with \( \varepsilon/2 \) replacing \( \varepsilon \), we deduce that

\[
J_5 \leq \sum_{n \geq (1+\varepsilon)\lambda(t)} \bar{F}(x)P(N(t) = n) \leq o(\bar{F}(x)\lambda(t)).
\]  

(3.21)

Now, substituting (3.18), (3.19), (3.20) and (3.21) into (3.17), we obtain (3.15). This ends the proof of Theorem 2.3.

### 3.4. Proof of Theorem 2.4

Clearly,

\[
P\left( \sup_{0 \leq t \leq T} \sum_{k=1}^{N(t)} (X_k - (1 + \delta)\mu) > x \right) \leq P\left( \sup_{0 \leq t \leq T} \sum_{k=1}^{N(t)} X_k > x \right)
\]

\[
= P\left( \sum_{k=1}^{N(T)} X_k > x \right)
\]

\[
= \lambda(T)\bar{F}(x),
\]

where, in the last step, we have used the classical inequality (3.1), Assumption D and the dominated convergence theorem (see for example Theorem A3.20 in Embrechts et al. (1997) for similar discussions). Now we aim to give the corresponding lower bound. For any large \( M > 0 \),

\[
P\left( \sup_{0 \leq t \leq T} \sum_{k=1}^{N(t)} (X_k - (1 + \delta)\mu) > x \right) \geq P\left( \sum_{k=1}^{N(T)} (X_k - (1 + \delta)\mu) > x \right)
\]

\[
\geq P\left( \sum_{k=1}^{N(T)} (X_k - (1 + \delta)\mu) > x, N(T) \leq M \right)
\]

\[
\geq \sum_{n=1}^{M} P(S_n > x + (1 + \delta)\mu M)P(N(T) = n).
\]
So it follows that, for the given $M > 0$,

$$
\liminf_{x \to \infty} \frac{P \left( \sup_{0 \leq t \leq T} \sum_{k=1}^{N(t)} (X_k - (1 + \delta)\mu) > x \right)}{\lambda(T) F(x)} \\
\geq \liminf_{x \to \infty} \sum_{n \leq M} \frac{P(S_n > x + (1 + \delta)\mu M) P(N(T) = n)}{\lambda(T) F(x)} \\
= \liminf_{x \to \infty} \sum_{n \leq M} \frac{P(\sum_{k=1}^{n} X_k > x + (1 + \delta)\mu M) P(N(T) = n)}{\lambda(T) F(x + (1 + \delta)\mu M)} \\
= \frac{E N(T) 1_{\{N(T) \leq M\}}}{\lambda(T)},
$$

where we have used the properties (2.1) and (2.2) of the subexponential class. Hence, by letting $M \to \infty$ on the right-hand side of the inequalities above, we also obtain the desired lower bound as $\lambda(T) F(x)$. This ends the proof of Theorem 2.4.

Acknowledgements

The authors would like to thank the referee for helpful suggestions and comments. This research was conducted while the second and the third authors were visiting the Department of Statistics and Actuarial Science, University of Hong Kong. They would like to thank the department for its kind hospitality. This work was supported by the Research Grants Council of HKSAR (project HKU 7139/01H), the Dutch Organization for Scientific Research (project NWO 42511013), the Ministry of Science and Technology of China (973 project on mathematics) and the Knowledge Innovation Programme of the Chinese Academy of Sciences.

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