UNIFORM ESTIMATES FOR THE TAIL PROBABILITY OF MAXIMA OVER FINITE HORIZONS WITH SUBEXPONENTIAL TAILS

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1. INTRODUCTION

Let $F$ be the common distribution function of the increments of a random walk $\{S_n, n \geq 0\}$ with $S_0 = 0$ and a negative drift and let $\{N(t), t \geq 0\}$ be a general counting process, independent of $\{S_n, n \geq 0\}$. This article investigates the tail probability, denoted by $c(\psi(x); t)$, of the maximum of $S_{N(t)}$ over a finite horizon $0 \leq v \leq t$. When $F$ is strongly subexponential, some asymptotics for $c(\psi(x); t)$ are derived as $x \to \infty$. The merit is that all of the obtained asymptotics are uniform for $t$ in a finite or infinite time interval.

Let $F$ be the common distribution function (d.f.) of the increments of a random walk $\{S_n, n \geq 0\}$ with $S_0 = 0$. This distribution function always satisfies $\bar{F}(x) = 1 - F(x) > 0$ for all $x$. We assume hereafter that $F$ has a finite mean $-\mu < 0$; hence, the random walk $S_n$ drifts to $-\infty$ and the ultimate maximum of the random walk is finite almost surely. Random walks with heavy-tailed increments and negative drifts have been extensively investigated in the literature; see, for example, Feller [9], Asmussen [1, 2], Embrechts, Klüppelberg, and Mikosch [7], and Rolski, Schmidli, Schmidt, and Teugels [21] for some background and important applications, see also a recent study by Denisov, Foss, and Korshunov [6] on an extreme case where the mean of the increment is not finite.
Veraverbeke [23] and Embrechts and Veraverbeke [8] investigated the asymptotic behavior of the tail probability of the ultimate maximum. Under the assumption that the d.f. \( F_{\infty} \) defined by (2.4) is subexponential, they provided a celebrated asymptotic relation that
\[
P \left( \max_{1 \leq n < \infty} S_n > x \right) = \frac{1}{\mu} \int_x^{\infty} \tilde{F}(y) \, dy. \tag{1.1} \]
Hereafter, any limit relationship is for \( x \to \infty \) unless otherwise stated; the symbol \( \sim \) means that the quotient of the left-hand side and right-hand side of the corresponding relationship tends to one in the indicated limiting sense. Relation (1.1) has been universally accepted as one of the most fundamental results in risk theory and some related fields. Korshunov [17] proved that the subexponentiality of \( F_{\infty} \) is also necessary for (1.1).

Throughout, let \( \{N(t), t \geq 0\} \) be a counting process, namely a nonnegative, nondecreasing, and integer-valued stochastic process, which is independent of the random walk \( \{S_n, n \geq 0\} \) and satisfies \( 0 < \lambda(t) = EN(t) < \infty \) for any \( t > 0 \). We write \( S(t) = S_{N(t)} \) for \( t \geq 0 \). In this article, we are interested in the tail probability of the maxima of the random sum \( S(t) \) over a finite time horizon. For this purpose, we introduce a bivariate function
\[
\psi(x; t) = \mathbb{P} \left( \max_{0 \leq v \leq x} S(v) > x \right) \quad \text{for} \quad x \geq 0 \text{ and } t \geq 0 \tag{1.2}
\]
and write
\[
\psi(x; \infty) = \lim_{t \to \infty} \psi(x; t) = \mathbb{P} \left( \max_{0 \leq v \leq x} S(v) > x \right) \quad \text{for} \quad x \geq 0. \tag{1.3}
\]
Since for any fixed \( x \geq 0 \) the function \( \psi(x; t) \) is nondecreasing in \( t \), the limit in (1.3) is well defined. Clearly, if \( \lambda(t) \to \infty \) as \( t \to \infty \), then (1.1) implies that the relation
\[
\psi(x; \infty) \sim \frac{1}{\mu} \int_x^{\infty} \tilde{F}(y) \, dy \tag{1.4}
\]
holds under the same assumptions.

According to Ng, Tang, Yan, and Yang [20], we propose an actuarial explanation for the functions \( \psi(x; t) \) and \( \psi(x; \infty) \). Rather than counting the number of claims, we count the number of customers buying insurance. The customer arrival process is assumed to be the counting process \( N(t), t \geq 0 \). After the \( k \)th customer has bought an insurance contract, \( k \geq 1 \), the insurance company will bear a risk from this policyholder within a fixed term. Suppose that the amount of all incurred claims due to the \( k \)th policyholder is \( Z_k \) and that \( Z_k, k \geq 1 \), are independent and identically distributed (i.i.d.) nonnegative random variables (r.v.’s) with finite mean. The premium paid for each policy is \( (1 + \delta)EZ_i \), where the positive constant \( \delta \) can be interpreted as the safety loading. Clearly, the insurance company’s total net risk due to the \( k \)th policyholder is
\[ X_k = Z_k - (1 + \delta)Z_{k-1}. \]

Assuming that the interest rate is zero, the total claim amount accumulated in the period \((0, t]\) can be written as
\[
\tilde{S}(t) = \sum_{k=1}^{N(t)} Z_k
\]
and the surplus of the company accumulated in the period \((0, t]\) can be written as
\[
U(t) = x - \sum_{k=1}^{N(t)} X_k = x - S(t),
\]
where \(x\) denotes the initial capital of the company. Thus, \(\psi(x; t)\) and \(\psi(x; \infty)\) defined by (1.2) and (1.3) represent the ruin probability within finite time \(t\) and the ultimate ruin probability, respectively.

In this article, we will investigate the asymptotic behavior of the tail probability \(\psi(x; t)\) and aim at asymptotic relations which are uniform for \(t\) in a finite or infinite time interval. As generally acknowledged by those experienced mathematicians, the uniformity often significantly merits the theoretical value of the asymptotic relations obtained. We remark that it is also crucial for many application problems; see Klüppelberg and Mikosch [14] and Embrechts et al. [7, Chap. 8] for related discussions.

Under the assumption that the d.f. \(F\) of the increments is strongly subexponential, applying Theorem 1 of Kaas and Tang [12], we can easily obtain that
\[
\psi(x; t) \sim \lambda(t)\bar{F}(x)
\]
holds for any fixed \(t > 0\); see also Foss and Zachary [10] for a similar result in a more general framework. In this article, we prove in Theorem 3.1 below that the asymptotic property in (1.6) is uniform for \(t\) in any finite interval \([t_1, t_2]\) for \(0 < t_1 \leq t_2 < \infty\); that is,
\[
\lim_{x \to \infty} \sup_{t_1 \leq t \leq t_2} \left| \frac{\psi(x; t)}{\lambda(t)\bar{F}(x)} - 1 \right| = 0.
\]
We remark that if \(\lambda(t) \to \infty\) as \(t \to \infty\), then the local uniformity of (1.6) cannot be extended to any infinite time interval. Based on some analysis, we intuitively believe that the asymptotic relation (1.6) will lose its accuracy when \(\lambda(t)\) gets large.

A significant large deviation result by Korshunov [18] makes it possible to derive an asymptotic relation for \(\psi(x; t)\) such that it is uniform for \(t\) in an infinite interval. We prove in Theorem 4.1 below that the uniformity for \(t \in [t_1, \infty)\) can be achieved by the asymptotic relation
\[
\psi(x; t) \sim \frac{1}{\mu} \int_x^{x+\mu\lambda(t)} \bar{F}(y) \, dy;
\]
that is,
\[
\lim_{x \to \infty} \sup_{t \in [t_1, t_2]} \left| \frac{\psi(x; t)}{\mu \int_x^{x+\mu(t)} \tilde{F}(y) \, dy} - 1 \right| = 0. \tag{1.9}
\]

This is the main contribution of the present article. In doing so, we only additionally assume that the counting process \(N(t), t \geq 0\) satisfies the law of large numbers. A closely related work in the literature is Klüppelberg and Mikosch [14], who successfully established a precise large deviation result for the random sum (1.5) as
\[
\lim_{t \to \infty} \sup_{x \in [y \Lambda(t), \infty)} \left| \frac{\mathbb{P}(\tilde{S}(t) - ES(t) > x)}{\lambda(t) \mathbb{P}(Z_1 > x)} - 1 \right| = 0 \tag{1.10}
\]
with \(y > 0\) arbitrarily fixed; see also the review article by Mikosch and Nagaev [19]. Clearly, the uniformity in (1.9) is more general than that in (1.10).

For the sake of completeness, we further consider the uniformity of the asymptotic relation (1.8) for \(t \in (0, \infty)\). We examine this uniformity for two special cases where \(N(t)\) are the Cox and renewal processes, respectively.

The outline of this article is as follows. Section 2 lists some preliminaries about heavy-tailed distributions; Section 3 proves that the asymptotic relation (1.6) is uniform for \(t\) in any finite interval \([t_1, t_2]\) for \(0 < t_1 \leq t_2 < \infty\); Section 4 proves that the asymptotic relation (1.8) holds uniformly for \(t \in [t_1, \infty)\) for any \(t_1 > 0\); Section 5 complements the study by considering the uniformity for \(t \in (0, \infty)\).

2. HEAVY-TAILED DISTRIBUTIONS

Like many recent researchers in the fields of applied probability and risk theory, we restrict our interest to the case of heavy-tailed d.f.’s. The most important class of heavy-tailed distributions is the subexponential class, denoted as \(\mathcal{S}\). By definition, a d.f. \(F\) concentrated on \([0, \infty)\) belongs to the class \(\mathcal{S}\) if and only if
\[
F^{\gamma^2}(x) \sim 2\bar{F}(x), \tag{2.1}
\]
where \(F^{\gamma^2}\) denotes the convolution of \(F\). More generally, a d.f. \(F\) concentrated on \((-\infty, \infty)\) is still said to be subexponential if the d.f. \(F^+(x) = F(x)1_{[0, \infty)}\) is subexponential, where and throughout \(1_A\) denotes the indicator function of a set \(A\). It is well known that if \(F \in \mathcal{S}\), then
\[
\bar{F}(x + l) \sim \bar{F}(x) \tag{2.2}
\]
for any fixed real number \(l\); see Chistyakov [5], or Embrechts et al. [7, Lemma 1.3.5]. Relation (2.2) characterizes another famous class of heavy-tailed distributions, de-
noted as \( L \). We easily check that for a d.f. \( F \in L \), there exists a positive function \( l_F(x) \to \infty \) such that

\[
\bar{F}(x + l_F(x)) \sim \bar{F}(x). \tag{2.3}
\]

A nice review on heavy-tailed distributions with applications to insurance and finance is the book by Embrechts et al. \[7\].

Recently, Korshunov \[18\] introduced another class of subexponential d.f.’s. For a subexponential d.f. \( F \) with \( m = \int_0^\infty \bar{F}(y) \, dy < \infty \), let

\[
\bar{F}_u(x) = \begin{cases} 
\min \left\{ 1, \int_x^{x+u} \bar{F}(y) \, dy \right\}, & x \geq 0 \\
1, & x < 0.
\end{cases} \tag{2.4}
\]

Clearly, for any \( u \in [0, \infty] \), \( F_u \) defines a standard d.f. concentrated on \([0, \infty)\). According to the terminology of Korshunov \[18\], we say that the d.f. \( F \) is strongly subexponential, denoted as \( F \in S_* \), if the relation

\[
\bar{F}_u(x) \sim 2\bar{F}(x)
\]

holds uniformly for \( u \in [1, \infty] \). From the discussions in Korshunov \[18\], we see that the class \( S_* \) almost coincides with the class of subexponential distributions with finite \( m = \int_0^\infty \bar{F}(y) \, dy \). Specifically, the class \( S_* \) contains the Pareto type (with parameter \( \alpha > 1 \)), the log-normal, and the Weibull distributions. It was also pointed out recently by Denisov, Foss, and Korshunov \[6, \text{Lemma 9}\] that another useful subexponential class \( S^{-} \), which was introduced by Klüppelberg \[13\] and is characterized by the property

\[
\int_0^x \bar{F}(x-y) \bar{F}(y) \, dy \sim 2m \bar{F}(x),
\]

is a subclass of the class \( S_* \). To the best of our knowledge, whether or not this inclusion \( S^{-} \subset S_* \) is strict remains unsolved.

We repeat here the result of Korshunov \[18\] for our quotation later:

**Lemma 2.1**: If the d.f. \( F \) of the increments of the random walk is strongly subexponential and has a finite mean \( -\mu < 0 \), then it holds uniformly for \( n \geq 1 \) that

\[
\mathbb{P} \left( \max_{1 \leq k \leq n} S_k > x \right) \sim \frac{1}{\mu} \int_x^{x+\mu \mu} \bar{F}(y) \, dy. \tag{2.5}
\]

Some intimately related results can be found in Borovkov and Borovkov \[4\].
3. ESTIMATE 1: \( t \in [t_1, t_2] \) FOR \( 0 < t_1 \leq t_2 < \infty \)

Recall that \( \{S_n, n \geq 0\} \) denotes a random walk with \( S_0 = 0 \) and a negative drift; that is, \( \{S_n - S_{n-1}, n \geq 1\} \) is a sequence of i.i.d. r.v.'s with common d.f. \( F \) and finite mean \(-\mu < 0\). Let \( N(t) \) be a nonnegative, nondecreasing, and integer-valued stochastic process, independent of the random walk \( \{S_n, n \geq 0\} \) and satisfying
\[
0 < \lambda(t) = \mathbb{E}N(t) < \infty \quad \text{for any } t > 0.
\]

In the sequel, for two positive infinitesimals \( A \sim x \) and \( B \sim x \), we write \( A \sim x \leq B \sim x \) if \( \lim \inf A \sim x / B \sim x \leq 1 \) and write \( A \sim x \geq B \sim x \) if \( \lim \sup A \sim x / B \sim x \leq 1 \). Clearly, \( A \sim x \leq B \sim x \) if and only if both \( A \sim x \leq B \sim x \) and \( A \sim x \geq B \sim x \).

The first result of this article is as follows:

**Theorem 3.1**: If \( F \in \mathcal{S} \), then for any \( 0 < t_1 \leq t_2 < \infty \), it holds uniformly for \( t \in [t_1, t_2] \) that
\[
\psi(x; t) \sim \lambda(t) \tilde{F}(x) \sim \frac{1}{\mu} \int_x^{x+\lambda(t)\mu} \tilde{F}(y) \, dy. \tag{3.1}
\]

We formulate the proof of Theorem 3.1 into a series of lemmas below, which have their own merits.

**Lemma 3.1**: If \( F \in \mathcal{S} \), then uniformly for \( 0 < t < \infty \), the relation
\[
\psi(x; t) \leq \lambda(t) \tilde{F}(x) \tag{3.2}
\]
holds; that is,
\[
\limsup_{x \to \infty} \sup_{0 < t < \infty} \frac{\psi(x; t)}{\lambda(t) \tilde{F}(x)} \leq 1.
\]

**Proof**: Since (2.5) holds uniformly for \( n \geq 1 \), we have uniformly for \( t \in (0, \infty) \) that
\[
\psi(x; t) = \sum_{n=1}^{\infty} \mathbb{P}\left( \max_{1 \leq k \leq n} S_k > x \right) \mathbb{P}(N(t) = n) \\
\sim \frac{1}{\mu} \sum_{n=1}^{\infty} \int_x^{x+\mu n} \tilde{F}(y) \, dy \mathbb{P}(N(t) = n) \tag{3.3}
\]
\[
\leq \lambda(t) \tilde{F}(x).
\]

This ends the proof of Lemma 3.1.

**Lemma 3.2**: If \( F \in \mathcal{L} \), then for any \( 0 < t_1 \leq t_2 < \infty \), the relation
\[
\psi(x; t) \geq \lambda(t) \tilde{F}(x) \tag{3.4}
\]
holds uniformly for \( t \in [t_1, t_2] \), where the uniformity is understood in a similar way as in Lemma 3.1.
Proof: Applying Lemma 1 of Korshunov [18] and recalling (2.3), in terms of the function \( I_F(x) \) in (2.3) we derive that, uniformly for \( t \in (0, \infty) \),

\[
\psi(x; t) = \frac{1}{\mu} \sum_{n=1}^{\infty} \int_x^{x+\mu} \bar{F}(y) \, dy \, \mathbb{P}(N(t) = n)
\]

\[
= \frac{1}{\mu} \sum_{1 \leq n \leq \mu^{-1} I_F(x)} \int_x^{x+\mu} \bar{F}(y) \, dy \, \mathbb{P}(N(t) = n)
\]

\[
= \sum_{1 \leq n \leq \mu^{-1} I_F(x)} n \bar{F}(x + I_F(x)) \mathbb{P}(N(t) = n)
\]

\[
= \lambda(t) \bar{F}(x) \left( 1 - \frac{1}{\lambda(t)} \mathbb{E} N(t) I_{(N(t) > \mu^{-1} I_F(x))} \right).
\]

(3.5)

In view that \( N(t) \), hence \( \lambda(t) \), is nondecreasing, we have

\[
\limsup_{x \to \infty} \sup_{t_1 \leq t \leq t_2} \frac{1}{\lambda(t)} \mathbb{E} N(t) I_{(N(t) > \mu^{-1} I_F(x))} \leq \frac{1}{\lambda(t_2)} \limsup_{x \to \infty} \mathbb{E} N(t_2) I_{(N(t_2) > \mu^{-1} I_F(x))} = 0.
\]

Substituting this into the right-hand side of (3.5), we obtain that

\[
\psi(x; t) \simeq \lambda(t) \bar{F}(x)
\]

holds uniformly for \( t_1 \leq t \leq t_2 \). This ends the proof of Lemma 3.2. \( \blacksquare \)

Lemma 3.3: If \( F \in \mathcal{L} \), then for any \( 0 < t_2 < \infty \), the relation

\[
\lambda(t) \bar{F}(x) \sim \frac{1}{\mu} \int_x^{x+\lambda(t) \mu} \bar{F}(y) \, dy
\]

(3.6)

holds uniformly for \( t \in (0, t_2] \).

Proof: Trivially, the inequality

\[
\lambda(t) \bar{F}(x) \geq \frac{1}{\mu} \int_x^{x+\lambda(t) \mu} \bar{F}(y) \, dy
\]
holds for all $0 < t, x < \infty$. On the other hand, recalling (2.2), we have, uniformly for $t \in (0, t_2]$, 
\[ \frac{1}{\mu} \int_x^{x + \lambda(t) \mu} \bar{F}(y) \, dy \geq \lambda(t) \bar{F}(x + \lambda(t) \mu) \]
\[ \geq \lambda(t) \bar{F}(x + \lambda(t_2) \mu) \]
\[ \sim \lambda(t) \bar{F}(x). \]

This ends the proof of Lemma 3.3. \[\square\]

4. ESTIMATE 2: $t \in [t_1, \infty)$ FOR $0 < t_1 < \infty$

This section is devoted to the uniform estimate for the probability $\psi(x; t)$ for large $t$.

Theorem 4.1: If the conditions

1. $F \in S_2$
2. $N(t)/\lambda(t) \to_p 1$ as $t \to \infty$

hold simultaneously, then for any $t_1 > 0$, it holds uniformly for $t \in [t_1, \infty)$ that

\[ \psi(x; t) \sim \frac{1}{\mu} \int_x^{x + \lambda(t) \mu} \bar{F}(y) \, dy. \] (4.1)

Proof: We bear in mind that all the asymptotic relations in the proof below hold uniformly for $t \in [t_1, \infty)$. The proof starts from the asymptotic relation (3.3):

\[ \psi(x; t) \sim \sum_{n=1}^{\infty} \frac{1}{\mu} \int_x^{x + \lambda(t) \mu} \bar{F}(y) \, dy \mathbb{P}(N(t) = n). \]

For arbitrarily fixed $0 < \delta < 1$, applying the techniques developed by Klüppelberg and Mikosch [14], we divide the series on the right-hand side into three parts as follows:

\[ \psi(x; t) \sim \sum_{1 \leq n < (1-\delta)\lambda(t)} + \sum_{(1-\delta)\lambda(t) \leq n \leq (1+\delta)\lambda(t)} + \sum_{n > (1+\delta)\lambda(t)} = I_1 + I_2 + I_3, \] (4.2)

Now, we treat the three terms respectively. Clearly,

\[ \frac{1}{\mu} \int_x^{x + \lambda(t) \mu} \bar{F}(y) \, dy \leq \mathbb{P}(N(t) < (1-\delta)\lambda(t)). \]
As for $I_3$, we have

$$\frac{1}{\mu} \int_x^{x+\lambda(t)\mu} \bar{F}(y) \, dy \leq \sum_{n>(1+\delta)\lambda(t)} \left( 1 + \frac{\int_x^{x+\lambda(t)\mu} \bar{F}(y) \, dy}{\int_x^{x+\lambda(t)\mu} \bar{F}(y) \, dy} \right) \mathbb{P}(N(t) = n)$$

$$= \sum_{n>(1+\delta)\lambda(t)} \frac{n}{\lambda(t)} \mathbb{P}(N(t) = n).$$

Then, we handle $I_2$:

$$I_2 = \sum_{(1-\delta)\lambda(t) < n < (1+\delta)\lambda(t)} \frac{1}{\mu} \int_x^{x+(1+\delta)\lambda(t)\mu} \bar{F}(y) \, dy \mathbb{P}(N(t) = n)$$

$$\leq \frac{1}{\mu} \int_x^{x+(1+\delta)\lambda(t)\mu} \bar{F}(y) \, dy \mathbb{P}((1-\delta)\lambda(t) \leq N(t) \leq (1+\delta)\lambda(t))$$

$$\leq \frac{1}{\mu} \int_x^{x+(1+\delta)\lambda(t)\mu} \bar{F}(y) \, dy. \quad (4.3)$$

Combining all these bounds together yields that

$$\psi(x;t) \leq \left( \mathbb{P}(N(t) < (1-\delta)\lambda(t)) + \sum_{n>(1+\delta)\lambda(t)} \frac{n}{\lambda(t)} \mathbb{P}(N(t) = n) \right)$$

$$\times \frac{1}{\mu} \int_x^{x+\lambda(t)\mu} \bar{F}(y) \, dy + \frac{1}{\mu} \int_x^{x+(1+\delta)\lambda(t)\mu} \bar{F}(y) \, dy. \quad (4.4)$$

By the assumption $N(t)/\lambda(t) \to_p 1$ as $t \to \infty$, we know that

$$\lim_{t \to \infty} \mathbb{P}(N(t) < (1-\delta)\lambda(t)) = 0.$$

Furthermore, from Lemma 3.1 in Ng, Tang, Yan, and Yang [20], we also have

$$\lim_{t \to \infty} \sum_{n>(1+\delta)\lambda(t)} \frac{n}{\lambda(t)} \mathbb{P}(N(t) = n) = 0.$$

Thus, for any $\varepsilon > 0$, there exists some large $t_\varepsilon = t_\varepsilon(\varepsilon, \delta) > t_1$ such that for all $t > t_\varepsilon$,

$$\mathbb{P}(N(t) < (1-\delta)\lambda(t)) + \sum_{n>(1+\delta)\lambda(t)} \frac{n}{\lambda(t)} \mathbb{P}(N(t) = n) \leq \varepsilon. \quad (4.5)$$
Substituting (4.5) into (4.4) yields

\[
\limsup_{x \to \infty} \sup_{t \geq t_0} \frac{\psi(x; t)}{\mu \int_x^{x + \Lambda(t) \mu} \Phi(y) \, dy} \leq \epsilon + \limsup_{x \to \infty} \sup_{t \geq t_0} \frac{\int_x^{x + \Lambda(t) \mu} \Phi(y) \, dy}{\mu \int_x^{x + \Lambda(t) \mu} \Phi(y) \, dy}
\]

\[
= 1 + \epsilon + \limsup_{x \to \infty} \sup_{t \geq t_0} \frac{\int_x^{x + \delta \Lambda(t) \mu} \Phi(y) \, dy}{\mu \int_x^{x + \Lambda(t) \mu} \Phi(y) \, dy}
\]

\[
\leq 1 + \epsilon + \limsup_{x \to \infty} \sup_{t \geq t_0} \frac{\delta \lambda(t) \mu \Phi(x + \lambda(t) \mu)}{\lambda(t) \mu \Phi(x + \lambda(t) \mu)}
\]

\[
= 1 + \epsilon + \delta.
\]

(4.6)

It follows that

\[
\limsup_{x \to \infty} \sup_{t \geq t_0} \frac{\psi(x; t)}{\mu \int_x^{x + \Lambda(t) \mu} \Phi(y) \, dy} = \limsup_{x \to \infty} \max \left\{ \sup_{t_1 \leq t \leq t_2} \frac{\psi(x; t)}{\mu \int_x^{x + \Lambda(t) \mu} \Phi(y) \, dy}, \sup_{t \geq t_0} \frac{\psi(x; t)}{\mu \int_x^{x + \Lambda(t) \mu} \Phi(y) \, dy} \right\}
\]

\[
= \max \left\{ \limsup_{x \to \infty} \sup_{t_1 \leq t \leq t_2} \frac{\psi(x; t)}{\mu \int_x^{x + \Lambda(t) \mu} \Phi(y) \, dy}, \limsup_{x \to \infty} \frac{\psi(x; t)}{\mu \int_x^{x + \Lambda(t) \mu} \Phi(y) \, dy} \right\}
\]

\[
\leq 1 + \epsilon + \delta,
\]

where, in the last step, we applied (4.6) and Theorem 3.1. By the arbitrariness of \( \epsilon \) and \( \delta \), we conclude that

\[
\limsup_{x \to \infty} \sup_{t \geq t_0} \frac{\psi(x; t)}{\mu \int_x^{x + \Lambda(t) \mu} \Phi(y) \, dy} \leq 1.
\]

(4.7)

Next, we turn to derive the lower bound for \( \psi(x; t) \). Using the same approach as applied in (4.3), we obtain the corresponding lower bound for \( I_2 \) that

\[
I_2 \geq \frac{1}{\mu} \int_x^{x + (1 - \delta) \lambda(t) \mu} \Phi(y) \, dy \mathbb{P}((1 - \delta) \lambda(t) \leq N(t) \leq (1 + \delta) \lambda(t)).
\]

(4.8)
The assumption $N(t)/\lambda(t) \to_p 1$ as $t \to \infty$ implies that there exists some large $t^* = t^*(\varepsilon, \delta) > t_1$ such that for all $t > t^*$,

$$\mathbb{P}((1 - \delta)\lambda(t) \leq N(t) \leq (1 + \delta)\lambda(t)) \geq 1 - \varepsilon. \quad (4.9)$$

Substituting (4.9) into (4.8) and recalling (4.2) and Theorem 3.1, we obtain

$$\liminf_{x \to \infty} \inf_{t \geq t^*} \frac{\psi(x; t)}{\mu \int_x^{x + \lambda(t)\mu} \bar{F}(y)\,dy}$$

$$= \liminf_{x \to \infty} \min_{t \geq t^*} \left\{ \inf_{t \geq t^*} \frac{\psi(x; t)}{\mu \int_x^{x + \lambda(t)\mu} \bar{F}(y)\,dy}, \inf_{t \geq t^*} \frac{\psi(x; t)}{\mu \int_x^{x + \lambda(t)\mu} \bar{F}(y)\,dy} \right\}$$

$$\geq \min \left\{ \liminf_{x \to \infty} \inf_{t \geq t^*} \frac{\psi(x; t)}{\mu \int_x^{x + \lambda(t)\mu} \bar{F}(y)\,dy}, \liminf_{x \to \infty} \frac{1}{\mu \int_x^{x + \lambda(t)\mu} \bar{F}(y)\,dy} \right\}$$

$$\geq \min \left\{ 1, (1 - \varepsilon) \liminf_{x \to \infty} \inf_{t \geq t^*} \frac{1}{\mu \int_x^{x + \lambda(t)\mu} \bar{F}(y)\,dy} \right\}$$

$$= \min \left\{ 1, (1 - \varepsilon) \left( 1 - \limsup_{x \to \infty} \inf_{t \geq t^*} \frac{\int_x^{x + \lambda(t)\mu} \bar{F}(y)\,dy}{\int_x^{x + \lambda(t)\mu} \bar{F}(y)\,dy} \right) \right\}$$

$$\geq \min \left\{ 1, (1 - \varepsilon) \left( 1 - \limsup_{x \to \infty} \inf_{t \geq t^*} \frac{\int_x^{x + (1 - \delta)\lambda(t)\mu} \bar{F}(y)\,dy}{\int_x^{x + (1 - \delta)\lambda(t)\mu} \bar{F}(y)\,dy} \right) \right\}$$

$$\geq \min \left\{ 1, (1 - \varepsilon) \left( 1 - \limsup_{x \to \infty} \inf_{t \geq t^*} \frac{\delta \lambda(t)\mu \bar{F}(x + (1 - \delta)\lambda(t)\mu)}{(1 - \delta)\lambda(t)\mu \bar{F}(x + (1 - \delta)\lambda(t)\mu)} \right) \right\}$$

$$= (1 - \varepsilon) \frac{1 - 2\delta}{1 - \delta}.$$
Finally, we conclude from (4.7) and (4.10) that

\[
\lim_{x \to \infty} \sup_{t \geq t_1} \left| \frac{\psi(x; t)}{\mu} \int_x^{x + \lambda(t) \mu} F(y) \, dy - 1 \right| = 0.
\]

This ends the proof of Theorem 4.1. □

5. ESTIMATE 3: \( t \in (0, \infty) \)

5.1. General Case

Now, we derive some uniform estimates for the tail probability \( \psi(x; t) \) over the time interval \((0, \infty)\). By virtue of Theorem 4.1, this amounts to considering the uniformity for \( t \in (0, 1] \). We assume that

\[
\lim_{x \to \infty} \sup_{0 < t \leq 1} \frac{1}{\lambda(t)} \mathbb{E} N(t) \mathbb{I}_{\{N(t) > x\}} = 0. \tag{5.1}
\]

Substituting (5.1) into the right-hand side of (3.5), going along the same line as the proof of Lemma 3.2, we obtain that

\[
\psi(x; t) \equiv \lambda(t) \bar{F}(x)
\]

holds uniformly for \( 0 < t \leq 1 \). Combining this with Lemmas 3.1 and 3.3 leads to the following lemma:

**Lemma 5.1:** If the conditions

1. \( F \in \mathcal{S}_s \)
2. \( N(t) \) satisfies (5.1)

hold simultaneously, then (3.1) holds uniformly for \( t \in (0, 1] \).

From Theorem 4.1 and Lemma 5.1, we immediately obtain the following theorem.

**Theorem 5.1:** If the conditions

1. \( F \in \mathcal{S}_s \)
2. \( N(t)/\lambda(t) \to_p 1 \) as \( t \to \infty \)
3. \( N(t) \) satisfies (5.1)

hold simultaneously, then it holds uniformly for \( t \in (0, \infty) \) that

\[
\psi(x; t) \sim \frac{1}{\mu} \int_x^{x + \lambda(t) \mu} \bar{F}(y) \, dy. \tag{5.2}
\]
5.2. Cox Case

Now, we seek some concrete examples such that assumption (5.1) is fulfilled. Let \( N_1(t), t \geq 0, \) be a homogeneous Poisson process with unit intensity and let \( \Lambda(t), t \geq 0, \) be another process independent of \( N_1(t) \) with the following properties:

1. \( \Lambda(0) = 0 \)
2. \( \mathbb{P}(\Lambda(t) < \infty) = 1 \) for any \( t \geq 0 \)
3. The trajectories of \( \Lambda(t) \) do not decrease and are right-continuous.

We say that a doubly stochastic process \( N(t) \) that is defined as the composition of \( N_1(\cdot) \) and \( \Lambda(\cdot) \),

\[
N(t) = N_1(\Lambda(t)), \quad t \geq 0,
\]  

is a Cox process. Clearly, it holds for any \( t > 0 \) that

\[
\Lambda(t) = \mathbb{E}N(t) = \mathbb{E}\Lambda(t).
\]

For overviews on Cox processes as well as their applications to actuarial and financial mathematics, we refer to Grandell [11] and Björk and Grandell [3]. Recent discussions on Cox processes can be found, for example, in Korolev [15, 16].

Lemma 5.2 shows that (5.1) can be satisfied by some Cox processes.

**Lemma 5.2:** Let \( N(t) \) be a Cox process with stochastic intensity \( \Lambda(t) \) satisfying

\[
\lim_{x \to \infty} \sup_{0 < t \leq 1} \frac{1}{\Lambda(t)} \mathbb{E}\Lambda(t)\mathbb{I}_{(\Lambda(t) > x)} = 0.
\]  

Then, \( N(t) \) satisfies assumption (5.1).

**Proof:** By the definition of Cox process, we derive

\[
\limsup_{x \to \infty} \sup_{0 < t \leq 1} \frac{1}{\Lambda(t)} \mathbb{E}N(t)\mathbb{I}_{(N(t) > x)} \leq \limsup_{x \to \infty} \sup_{0 < t \leq 1} \frac{1}{\Lambda(t)} \mathbb{E} \left( \sum_{n=1}^{\infty} \frac{\Lambda^n(t)}{n!} e^{-\Lambda(t)} \right)
\]

\[
= \limsup_{x \to \infty} \sup_{0 < t \leq 1} \frac{1}{\Lambda(t)} \times \mathbb{E}(\Lambda(t)\mathbb{P}(N(t) > x - 1|\Lambda(t))).
\]

For arbitrarily fixed \( M > 0 \), we continue the derivation as follows:

\[
\limsup_{x \to \infty} \sup_{0 < t \leq 1} \frac{1}{\Lambda(t)} \mathbb{E}N(t)\mathbb{I}_{(N(t) > x)}
\]

\[
\leq \limsup_{x \to \infty} \sup_{0 < t \leq 1} \frac{1}{\Lambda(t)} \mathbb{E}(\Lambda(t)\mathbb{P}(N(t) > x - 1|\Lambda(t)) [\mathbb{I}_{(\Lambda(t) > M)} + \mathbb{I}_{(\Lambda(t) \leq M)}])
\]

\[
\leq \limsup_{x \to \infty} \sup_{0 < t \leq 1} \frac{1}{\Lambda(t)} \left[ \mathbb{E}(\Lambda(t)\mathbb{I}_{(\Lambda(t) > M)}) + \mathbb{E}\Lambda(t)\mathbb{P}(N_1(M) > x - 1) \right]
\]

\[
= \sup_{0 < t \leq 1} \frac{1}{\mathbb{E}\Lambda(t)} \mathbb{E}(\Lambda(t)\mathbb{I}_{(\Lambda(t) > M)}).
\]
By assumption (5.4), we know that the right-hand side of the above vanishes as $M \to \infty$. Therefore, (5.1) is fulfilled. This ends the proof of Lemma 5.2.

Based on Lemma 5.2, the following result is an immediate consequence of Theorem 5.1.

**Corollary 5.1:** Let $N(t)$ be a Cox process with stochastic intensity $\Lambda(t)$ satisfying (5.4). If the conditions

1. $F \in S_c$
2. $\Lambda(t)/\mathbb{E}\Lambda(t) \to_p 1$ as $t \to \infty$

hold simultaneously, then (5.2) holds uniformly for $t \in (0, \infty)$.

**Proof:** By Lemma 5.2, assumption (5.4) implies assumption (5.1). Furthermore, by Lemma 2 in Korolev [16], we see that as $t \to \infty$, $\Lambda(t)/\mathbb{E}\Lambda(t) \to_p 1$ implies $N(t)/\lambda(t) \to_p 1$. Hence, Theorem 5.1 gives the result in Corollary 5.1.

We remark that $N(t)$ in Corollary 5.1 can be specified as any homogeneous or inhomogeneous Poisson process.

**5.3. Renewal Case**

The second case we consider is the renewal process $N(t)$ defined by

$$N(t) = \max\left\{ n \geq 1 : \sum_{k=1}^{n} Y_k \leq t \right\}, \quad t \geq 0,$$

(5.5)

where $\{Y_k, k \geq 1\}$ is a sequence of i.i.d. and nonnegative r.v.'s and $\max \phi = 0$ by convention. In order that $\lambda(t) > 0$ for any $t > 0$, we assume

$$\mathbb{P}(Y_1 = 0) < 1, \quad \mathbb{P}(0 \leq Y_1 < \varepsilon) > 0 \quad \text{for any } \varepsilon > 0.$$

(5.6)

**Lemma 5.3:** Consider the renewal process $N(t)$ defined in (5.5). If $Y_1$ satisfies assumption (5.6), then $N(t)$ satisfies assumption (5.1).

**Proof:** By condition (5.6), there exists some integer $m_0 \geq 1$ such that $\mathbb{P}(\sum_{k=1}^{m_0} Y_k \leq 1) < 1$ holds. We write

$$p_0(t) = \mathbb{P}\left(\sum_{k=1}^{m_0} Y_k \leq t\right) \quad \text{for } 0 < t \leq 1.$$

Clearly, $0 < p_0(t) \leq p_0(1) < 1$ for each $0 < t \leq 1$. Hence, for any $n \geq 1$, we derive that
This proves that assumption (5.1) holds. 

By the well-known renewal theory, we know that if the r.v. $Y_1$ in (5.5) has a finite mean $1/\lambda > 0$, then

$$\frac{N(t)}{\lambda(t)} \to_p 1, \quad \lambda(t) \sim \lambda t \quad \text{as } t \to \infty; \quad (5.7)$$

see, for example, Feller [9, Chap. XI]. Based on (5.7) and Lemma 5.3, the following result is another immediate consequence of Theorem 5.1.

**Corollary 5.2**: Let $N(t)$ be a renewal process defined by (5.5). If the conditions

1. $F \in S$
2. $Y_1$ satisfies (5.6) and has a finite mean $1/\lambda$

hold simultaneously, then (5.2) holds uniformly for $t \in (0, \infty)$. Furthermore, for any positive function $f(x) \to \infty$, it holds uniformly for $t \geq f(x)$ that

$$\psi(x; t) \sim \frac{1}{\mu} \int_x^{x+\lambda t} \tilde{F}(y) dy;$$

that is,

$$\lim_{x \to \infty} \sup_{t \geq f(x)} \left| \frac{\psi(x; t)}{\frac{1}{\mu} \int_x^{x+\lambda t} \tilde{F}(y) dy} - 1 \right| = 0.$$
Acknowledgments
The author would like to thank Professor Rob Kaas for his helpful discussions. This work was supported by the Dutch Organization for Scientific Research (project no. NWO 42511013).

References