The Ruin Probability of a Discrete Time Risk Model under Constant Interest Rate with Heavy Tails

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This paper investigates the ultimate ruin probability of a discrete time risk model with a positive constant interest rate. Under the assumption that the gross loss of the company within one year is subexponentially distributed, a simple asymptotic relation for the ruin probability is derived and compared to existing results. Key words: Asymptotics, constant interest rate, Matuszewska index, ruin probability, subexponentiality.

1. INTRODUCTION

In this paper we are interested in the ruin probability of a discrete time risk model with a constant interest rate. In this model the surplus process of the company is expressed by the recursive equation

\[ U_0 = x, \quad U_n = U_{n-1}(1 + r) - X_n, \quad n = 1, 2, \ldots, \]  

where \( x \geq 0 \) is the initial surplus, \( r \geq 0 \) is the constant interest rate, \( X_n \) denotes the gross loss (i.e. the total claim amount minus the total incoming premium) during the \( n \)th year, and \( X_n, n = 1, 2, \ldots, \) constitute a sequence of independent, identically distributed (i.i.d.), and real-valued random variables (r.v.'s) with generic r.v. \( X \) and common distribution function (d.f.) \( F(x) = 1 - \mathbb{P}(X \leq x) \) for \( x \in (-\infty, \infty) \). The ultimate ruin probability is then defined as

\[ \psi_r(x) = \mathbb{P}\left( \min_{0 \leq n < \infty} U_n < 0 \mid U_0 = x \right), \quad x, r \geq 0. \]  

When \( r = 0 \) this model corresponds to the discrete time renewal risk model (Sparre Andersen model). In the latter model, if \( -\mu = \mathbb{E}X < 0 \) and the equilibrium d.f. of \( F \) is subexponential (see Section 2 for the definition), a celebrated asymptotic relation was established by Veraverbeke (1977, Theorem 2(B)), showing that as \( x \to \infty \),

\[ \psi_0(x) \sim \frac{1}{\mu} \int_x^\infty F(y) \, dy; \]  

see also Embrechts & Veraverbeke (1982) for the treatment in the continuous time model.

The purpose of this paper is to derive a simple asymptotic formula for the ruin probability \( \psi_r(x) \) for \( r > 0 \). We remark that in the recent literature the ruin probability under a constant interest force but in a continuous time risk model (namely the
compound Poisson model) has been extensively investigated by a series of papers such as Sundt & Teugels (1995, 1997), Klüppelberg & Stadtmüller (1998), Asmussen (1998), and Kalashnikov & Konstantinides (2000), among others. Specifically, in our recent work Konstantinides et al. (2002), when establishing two-sided bounds for the ruin probability of the mentioned continuous time risk model, we introduced the following mild condition:

$$\limsup_{x \to \infty} \frac{F(xy)}{F(x)} < 1$$

for some $y > 1$. (1.4)

The investigation there shows that it is necessary to assume restriction (1.4) on the equilibrium distribution of the common claim size in order to build an asymptotic formula similar to (1.3) for the ruin probability.

The standing assumption of this paper is that the common d.f. $F$ is subexponential and satisfies (1.4). The latter assumption will play a crucial role in our derivation. By the recursive equation (1.1) we know that for $x > 0$,

$$U_0 = x, \quad U_n = x(1 + r)^n - \sum_{k=1}^{n} X_k (1 + r)^{n-k}, \quad n = 1, 2, \ldots.$$

Hence, the ruin probability $\psi_r(x)$ can be rewritten in terms of the discounted values of the surplus process as

$$\psi_r(x) = \mathbb{P} \left( \min_{0 \leq n < \infty} (1 + r)^{-n} U_n < 0 \mid U_0 = x \right) = \mathbb{P} \left( \max_{0 \leq n < \infty} \sum_{k=0}^{n} X_k (1 + r)^{-k} > x \right).$$

(1.5)

where $X_0 = 0$ by convention. As a necessary step in modelling, we will address in Lemma 4.3 below that the maximum in (1.5) is a proper r.v. concentrated on $[0, \infty)$ if $r > 0$ and (1.4) is fulfilled, regardless of whether or not the safety loading condition holds; hence, the ruin probability $\psi_r(x)$ in (1.2) is well defined. Our approach, which applies techniques developed by Resnick (1987) and Davis & Resnick (1988), is different from those used in Klüppelberg & Stadtmüller (1998), Asmussen (1998), and Kalashnikov & Konstantinides (2000).

For related recent work on the ruin probability of discrete time models with a constant or stochastic interest rate, we refer to Yang (1999), Nyrhinen (1999, 2001), Cai (2002) and Tang & Tsitsiashvili (2003), among many others.

The rest of this paper is organized as follows: Section 2 recalls some preliminaries on heavy-tailed distributions, Section 3 presents the main results, Section 4 establishes some lemmas (most of which are of independent interest on their own right), and Section 5 gives the proofs of the main results.

2. HEAVY-TAILED DISTRIBUTIONS

Hereafter, all limit relationships are for $x \to \infty$ unless stated otherwise. For two positive infinitesimals $A(x)$ and $B(x)$, we write $A(x) \lesssim B(x)$ if $\limsup A(x)/B(x) \leq 1$, $A(x) \gtrsim B(x)$ if $\liminf A(x)/B(x) \geq 1$, and write $A(x) \sim B(x)$ if both.
A r.v. \( X \) or its d.f. \( F \) satisfying \( F(x) > 0 \) for all \( x \in (-\infty, \infty) \) is heavy tailed to the right if \( \mathbb{E}e^{xX} = \infty \) for all \( y > 0 \). One of the most important classes of heavy-tailed distributions is the subexponential class. By definition, a d.f. \( F \) concentrated on \([0, \infty)\) is subexponential, denoted by \( F \in \mathcal{S} \), if and only if

\[
\lim_{x \to \infty} \frac{F^2(x)}{F(x)} = 2, \tag{2.1}
\]

where \( F^2 \) denotes the convolution of \( F \). More generally, a d.f. \( F \) concentrated on \((-\infty, \infty)\) is still said to be subexponential to the right if the d.f. \( F^+(x) = F(x I_{\{0 \leq x < \infty\}}) \) is subexponential. It is easy to verify that (2.1) remains valid for the latter general case; see also Lemma 4.4 below. It is well known that if \( F \in \mathcal{S} \) then

\[
\lim_{x \to \infty} \frac{F(x + y)}{F(x)} = 1 \tag{2.2}
\]

for any fixed real number \( y \); see Embrechts et al. (1997, Lemma 1.3.5) and the references therein. Relation (2.2) characterizes the class of long-tailed distributions, denoted by \( \mathcal{L} \). A famous subclass of subexponential distributions is the class \( \mathcal{R} \) of d.f.’s with regularly varying tails. By definition, a d.f. \( F \) concentrated on \((-\infty, \infty)\) belongs to the class \( \mathcal{R} \) if and only if there exist some constant \( 0 < a < \infty \) and positive slowly varying function \( L(x) \) such that

\[
F(x) = x^{-a} L(x). \tag{2.3}
\]

We denote this case by \( F \in \mathcal{R}_a \). For reviews on heavy-tailed distributions and their applications to insurance and finance, the readers are referred to Bingham et al. (1987) and Embrechts et al. (1997).

The following class complements the class \( \mathcal{R} \) with an extreme case:

**DEFINITION 2.1.** Let \( F \) be a d.f. concentrated on \((-\infty, \infty)\). \( F \) is said to be rapidly varying tailed, denoted by \( F \in \mathcal{R}_{-\infty} \), if and only if

\[
\lim_{x \to \infty} \frac{F(xy)}{F(x)} = 0, \quad \text{for each} \quad y > 1. \tag{2.4}
\]

We remark that a more general notion, rapid variation, has been investigated in the literature; see the monographs Geluk & de Haan (1987) and Bingham et al. (1987).

We recall here a significant index of distributions, which is crucial for our purpose. Let \( F \) be concentrated on \((-\infty, \infty)\). For any \( y > 0 \) we set

\[
\overline{F}(y) = \limsup_{x \to \infty} \frac{F(xy)}{F(x)}
\]

and then define
\[ J_F^- = \sup \left\{ -\frac{\log F^*(y)}{\log y} : y > 1 \right\} = -\lim_{y \to \infty} \frac{\log F^*(y)}{\log y}. \] (2.5)

In the terminology of Bingham et al. (1987), the quantity \( J_F^- \) are the lower Matuszewska index of the function \( f = 1/F \). Without any confusion we simply call \( J_F^- \) the lower Matuszewska index of the d.f. \( F \). The latter equality in (2.5) is due to Theorem 2.1.5 in Bingham et al. (1987). For more details of the Matuszewska indices, see Bingham et al. (1987, Chapter 2.1), Cline and Samorodnitsky (1994), as well as Tang et al. (2003). Clearly, if the d.f. \( F \in \mathcal{R}_- \) for some \( 0 < x < \infty \) then \( J_F^- = x \); the d.f. \( F \) has a lower Matuszewska index \( 0 < J_F^- \leq \infty \) if and only if it satisfies restriction (1.4).

In the recent work Konstantinides et al. (2002), we introduced a new subclass, \( \mathcal{A} \), of subexponential d.f.’s. In terms of the lower Matuszewska index we can restate its definition as follows:

**DEFINITION 2.2.** Let \( F \) be a d.f. concentrated on \( (-\infty, \infty) \). We say that \( F \) belongs to the class \( \mathcal{A} \) if \( F \) is subexponential and has a lower Matuszewska index \( 0 < J_F^- \leq \infty \).

The purpose that we introduced this class is mainly to exclude some very heavy-tailed (like slowly varying tailed) distributions from the class \( \mathcal{S} \). It is easy to see that

\[ \mathcal{R}_- \subset \mathcal{A} \quad \text{for} \quad 0 < x < \infty, \quad \mathcal{S} \cap \mathcal{R}_- \subset \mathcal{A}. \]

Hence, the class \( \mathcal{A} \) covers almost all subexponential d.f.’s of interest. Actually, any subexponential d.f. \( F \) is an element of the class \( \mathcal{A} \) if its right-hand side behaves like the Pareto, the Lognormal, the Weibull, the Loggamma, the Burr, the Benktander I or II distribution; see Embrechts et al. (1997, Table 1.2.6) and Asmussen (1998).

3. MAIN RESULTS AND REMARKS

The main contribution of this paper is the following result:

**THEOREM 3.1.** Suppose that \( F \in \mathcal{A} \). Then it holds that

\[ \psi_r(x) \sim \sum_{k=1}^{\infty} F((1 + r)^k x), \quad r > 0. \] (3.1)

By inequality (4.2) below it is easily seen that the series in (3.1) converges for arbitrarily fixed \( x > 0 \). By inequality (4.1) below, it holds for some \( C > 0 \) that

\[ F((1 + r)x) \leq \sum_{k=1}^{\infty} F((1 + r)^k x) \leq CF((1 + r)x). \]
Hence, the ruin probability $\psi_c(x)$ has the same order as $\overline{F}(1 + r)x$. We put forward two special cases of Theorem 3.1 as follows, in which more accurate approximations are presented:

COROLLARY 3.1. If $F \in \mathcal{H}_{-c}$ for some $0 < x < \infty$, then it holds that

$$
\psi(x) \sim \frac{1}{(1 + r)^x - 1} \overline{F}(x), \quad r > 0. 
$$

(3.2)

If $F \in \mathcal{F} \cap \mathcal{H}_{-x}$, then it holds that

$$
\psi_c(x) \sim \overline{F}(1 + r)x, \quad r > 0. 
$$

(3.3)

REMARK 3.1. Let $Y_n$ and $Z_n$ denote the total incoming premium and total claim amount during the $n$th year, respectively, $n = 1, 2, \ldots$. We simply assume that \{\(Y_n, n = 1, 2, \ldots\)\} and \{\(Z_n, n = 1, 2, \ldots\)\} are sequences of i.i.d. non-negative r.v.'s, that the two sequences are mutually independent, and that $X_n = Z_n - Y_n$, $n = 1, 2, \ldots$.

As is well known in the literature (see also Lemma 4.2 below), within the class $\mathcal{L}$, the relation $P(X_1 > x) \sim P(Z_1 > x)$ holds. Hence, Theorem 3.1 and Corollary 3.1 remain valid if we rewrite them in terms of the d.f. $G$ of $Z_1$. 

REMARK 3.2. Now we relate Theorem 3.1 to the continuous time model. In this model the claim sizes, $Z_k, k = 1, 2, \ldots$, form a sequence of i.i.d. non-negative r.v.'s; their arrival times, $T_k, k = 1, 2, \ldots$, constitute a homogeneous Poisson process, which is independent of \{\(Z_k, k = 1, 2, \ldots\)\}; the total claim amount accumulated up to time $t \geq 0$ is

$$
Z(t) = \sum_{k=1}^{\infty} Z_k I(T_k \leq t);
$$

the premium rate and interest force are two positive constants $c$ and $\delta$, respectively. With the initial surplus $x \geq 0$, the total surplus process is described by the stochastic equation

$$
U(t) = xe^{\delta t} + c \int_0^t e^{\delta(t-y)} dy - \int_0^t e^{\delta(t-y)}Z(dy), \quad t \geq 0;
$$

(3.4)

see Sundt & Teugels (1995, 1997). If we denote by $X_k$ the gross loss during the $k$th year, $k = 1, 2, \ldots$, then it is easily understood that

$$
X_k = \int_{k-1}^{k} e^{\delta(k-y)}Z(dy) - c \int_{k-1}^{k} e^{\delta(k-y)} dy = \sum_{i=1}^{\infty} Z_i e^{\delta(k-T_i)}I_{(k-1,T_i,k)} - c \int_0^1 e^{\delta y} dy.
$$

(3.5)

Hence, $X_k, k = 1, 2, \ldots$, also constitute a sequence of i.i.d. r.v.'s because of the stationarity of the sequence \{\(T_k, k = 1, 2, \ldots\)\}. With notation $r = e^{\delta} - 1 > 0$, the
embedded process $U_n = U(n)$, $n = 0, 1, \ldots$, is recognized as satisfying the recursive equation (1.1). So the ruin probability defined by (1.2) gives a lower bound for the ruin probability of the compound Poisson model (3.4); see Kalashnikov & Norberg (2002) for related discussions.

**Remark 3.3.** The interest earned by the company can only help, of course, to reduce the ruin probability. Now we illustrate how strong this influence is on the solvency of the company. Observe formulae (1.3) and (3.2). When $F \in \mathcal{R}_{-\infty}$ (or equivalently, $G \in \mathcal{R}_{-\infty}$) for some $1 < \alpha < \infty$, applying Karamata’s theorem and taking into account representation (2.3) for the regularly varying tail, it is easy to check that

$$
\psi_0(x) \sim \frac{(1 + r)^x - 1}{(x - 1)\mu} x \psi_1(x).
$$

This shows that, as remarked by Klüppelberg and Stadtmüller (1998), the interest earned by the company reduces the order of the ruin probability by one power of $x$.

For Karamata’s theorem we refer to Proposition 1.5.10 of Bingham et al. (1987) and Theorem A3.6 of Embrechts et al. (1997). Stronger influence of the positive interest can be observed if $F \in \mathcal{Y} \cap \mathcal{R}_{-\infty}$ (or equivalently, $G \in \mathcal{Y} \cap \mathcal{R}_{-\infty}$). Actually, for this case we can derive from formulae (1.3) and (3.3) that

$$
\limsup_{x \to \infty} \frac{x \psi_1(x)}{\psi_0(x)} \leq \limsup_{x \to \infty} \frac{x \mathcal{F}(1 + r)x}{\mu^{-1} \int_x^{(1 + r/2)x} \mathcal{F}(y) dy} \leq \limsup_{x \to \infty} \frac{2\mu \mathcal{F}(1 + r)x}{r \mathcal{F}((1 + r/2)x)} = 0.
$$

If the d.f. $G$ is the Weibull or the Benktander II distribution, we can even obtain that $x^M \psi_1(x) = o(\psi_0(x))$ for any $M > 0$. □

**4. Lemmas**

**Lemma 4.1.** Let $F$ be a d.f. concentrated on $(-\infty, \infty)$ with a lower Matuszewska index $0 < \underline{\beta}_F \leq \infty$. Then for any $0 < \alpha < \underline{\beta}_F$ there are positive constants $C_1$, $C_2$, and $x_0$ such that the inequality

$$
\frac{\mathcal{T}(xy)}{\mathcal{T}(x)} \leq C_1 y^{-\alpha}
$$

holds uniformly for $xy \geq x \geq x_0$, and that the inequality

$$
\mathcal{T}(x) \leq C_2 x^{-\alpha}
$$

holds uniformly for $x \geq x_0$.

**Proof.** Since $\underline{\beta}_F$ is actually the lower Matuszewska index of the function $f = 1/\mathcal{T}$, some simple adjustment on the second inequality of Proposition 2.2.1 in Bingham et al. (1987) gives inequality (4.1). Fixing the variable $x = x_0$ in (4.1) yields that the inequality

$$
\frac{\mathcal{T}(xy)}{\mathcal{T}(x)} \leq C_1 y^{-\alpha}
$$

holds uniformly for $xy \geq x \geq x_0$. □
\[
\frac{F(x_0 y)}{F(x_0)} \leq C_1 y^{-z}
\]

holds uniformly for \( y \geq 1 \). Then, substitution \( x = x_0 y \) leads to (4.2). \qed

Now we consider the difference
\[ X = Z - Y, \quad (4.3) \]
where \( Z \) and \( Y \) are independent r.v.’s, and \( Y \) is non-negative and non-degenerate at 0. Let \( X, Y, \) and \( Z \) be distributed by \( F_X, F_Y, \) and \( F_Z \), respectively.

**Lemma 4.2.** Consider (4.3). We have
\[ F_X \in \mathcal{L} \Leftrightarrow F_Y \in \mathcal{L} \Leftrightarrow F_X(x) \sim F_Z(x). \]

**Proof.** The implication “\( F_Z \in \mathcal{L} \Rightarrow F_X(x) \sim F_Z(x) \)” is well known and follows from the dominated convergence theorem. To our knowledge the other implications are new.

First we prove the implication “\( F_Z \in \mathcal{L} \Rightarrow F_X \in \mathcal{L} \)”. Since the r.v. \( Y \) is non-negative, we have that, for each \( M > 0, \ y > 0 \) and \( x \geq M \),
\[ \{ Z > x, \ Y \leq M \} \subset \{ X > x - M \}, \quad \{ X > x + y \} \subset \{ Z > x + y \}. \]
Hence, \( F_Z(x)F_Y(M) \leq F_X(x - M) \) and \( F_X(x + y) \leq F_Z(x + y) \). It follows that
\[
\frac{F_Z(x + y)}{F_Z(x)} \geq \frac{F_X(x + y)}{F_X(x - M)} F_Y(M).
\]
Since \( F_X \in \mathcal{L} \), first taking \( x \to \infty \) then taking \( M \to \infty \) on the above inequality leads to
\[
\liminf_{x \to \infty} \frac{F_Z(x + y)}{F_Z(x)} \geq 1,
\]
which implies \( F_Z \in \mathcal{L} \).

Next we prove the implication “\( F_X(x) \sim F_Z(x) \Rightarrow F_X \in \mathcal{L} \)”. We choose some \( y_0 > 0 \) such that \( F_Y(y_0) > 0 \). Clearly, it holds that
\[
F_X(x) \leq F_Z(x)F_Y(y_0) + F_Z(x + y_0)F_Y(y_0).
\]
Therefore
\[
1 \leq F_Y(y_0) + F_Y(y_0) \liminf_{x \to \infty} \frac{F_Z(x + y_0)}{F_Z(x)}
\]
which implies that
\[
\liminf_{x \to \infty} \frac{F_Z(x + y_0)}{F_Z(x)} \geq 1.
\]
This proves \( F_Z \in \mathcal{L} \), hence \( F_X \in \mathcal{L} \). \qed
LEMMA 4.3. With the notation introduced in Section 1, the maximum

$$\max_{0 < n < \infty} \sum_{k=0}^{n} X_k(1 + r)^{-k}$$

is a proper r.v. concentrated on \([0, \infty)\) if \(r > 0\) and \(0 < J_F^- \leq \infty\).

**Proof.** Observe that

$$0 \leq \max_{0 < n < \infty} \sum_{k=0}^{n} X_k(1 + r)^{-k} \leq \sum_{k=1}^{\infty} X_k^+(1 + r)^{-k}, \quad (4.4)$$

where \(X_k^+ = X_k I_{X_k > 0}\), \(k = 1, 2, \ldots\). It suffices to verify the a.s. convergence of the infinite series in (4.4). To this end, we notice that inequality (4.2) implies

$$\mathbb{E}(X^+)^2 < \infty \quad \text{for all} \quad 0 < x < \min\{J_F^-, 1\}. \quad \text{Hence,}$$

$$\mathbb{E}\left(\sum_{k=1}^{\infty} X_k^+(1 + r)^{-k}\right)^2 \leq \sum_{k=1}^{\infty} \mathbb{E}(X_k^+(1 + r)^{-k})^2 < \infty.$$  

This ends the proof. \(\square\)

The following result can be obtained by fixing \(\gamma = 0\) in Lemma 3.2 of Tang et al. (2003). Under some additional restrictions this result was first established by Embrechts & Goldie (1980) and Cline (1986, Corollary 1).

LEMMA 4.4. Let \(F = F_1 \ast F_2\), where \(F_1\) and \(F_2\) are two d.f.'s concentrated on \((-\infty, \infty)\). If \(F_1 \in \mathcal{S}, F_2 \in \mathcal{L},\) and \(\mathcal{F}_2(x) = O(\mathcal{F}_1(x))\), then \(F \in \mathcal{S}\) and

$$\mathcal{F}(x) \sim \mathcal{F}_1(x) + \mathcal{F}_2(x). \quad (4.5)$$

Now we establish a result similar to (4.5) but in an inequality form.

LEMMA 4.5. Let \(F = F_1 \ast F_2\), where \(F_1\) and \(F_2\) are two d.f.'s concentrated on \((-\infty, \infty)\). If \(F_1 \in \mathcal{S}\) and \(\mathcal{F}_2(x) \leq c\mathcal{F}_1(x)\) for some \(c \geq 0\), then

$$\mathcal{F}(x) \leq (1 + c)\mathcal{F}_1(x). \quad (4.6)$$

**Proof.** For any \(\varepsilon > 0\) we choose some \(M > 0\) such that \(\mathcal{F}_2(x) \leq (c + \varepsilon)\mathcal{F}_1(x)\) holds for all \(x \geq M\). Relying on this \(M\) we derive.

$$\mathcal{F}(x) = \left(\int_{-\infty}^{x-M} + \int_{x-M}^{\infty}\right) \mathcal{F}_2(x-y)\mathcal{F}_1(dy) \leq (c + \varepsilon) \int_{-\infty}^{x} \mathcal{F}_1(x-y)\mathcal{F}_1(dy) + \mathcal{F}_1(x) \quad (4.7)$$

where we have used the condition \(F_1 \in \mathcal{S}\). Some direct calculation gives the equality

$$\int_{-\infty}^{x} \mathcal{F}_1(x-y)\mathcal{F}_1(dy) = F_1^{2\ast}(x) - \mathcal{F}_1(0)\mathcal{F}_1(x) - \int_{-\infty}^{0} \mathcal{F}_1(x-y)\mathcal{F}_1(dy), \quad (4.8)$$
The condition $F_1 \in \mathcal{S}$ implies $F_1^2(x) \sim 2F_1(x)$; the dominated convergence theorem gives $\int_{-\infty}^{\infty} F_1(x-y) F_1(dy) \sim F_1(0)F_1(x)$. Hence, it follows from (4.8) that
\[
\int_{-\infty}^{\infty} F_1(x-y) F_1(dy) \sim F_1(x). \tag{4.9}
\]
Substituting (4.9) into (4.7) yields that $F(x) \leq (1 + c + \varepsilon)F_1(x)$. Relation (4.6) follows from the arbitrariness of $\varepsilon > 0$.

The following is the restatement of Theorem 2.1 in Ng et al. (2002).

**Lemma 4.6.** Let $\{X_k, k = 1, 2, \ldots\}$ be a sequence of independent r.v.’s. If each $X_k$ has a d.f. $F_k \in \mathcal{S}$, $k = 1, 2, \ldots$, then it holds for each $m = 1, 2, \ldots$ that
\[
P\left(\max_{1 \leq n \leq m} \sum_{k=1}^{n} X_k > x\right) \sim P\left(\sum_{k=1}^{m} X_k > x\right). \tag{4.10}
\]

**Lemma 4.7.** Let $\{X_k, k = 1, 2, \ldots\}$ be a sequence of i.i.d. r.v.’s with common d.f. $F \in \mathcal{S}$, and let $\{c_k, k = 1, 2, \ldots\}$ be a sequence of positive constants. Then, for each $n = 1, 2, \ldots$, the finite weighted sum $\sum_{k=1}^{n} c_k X_k$ is subexponentially distributed and
\[
P\left(\sum_{k=1}^{n} c_k X_k > x\right) \sim \sum_{k=1}^{n} F(x/c_k). \tag{4.11}
\]

*Proof.* Clearly, by virtue of Lemma 4.4 we can prove (4.11) for $n = 2$. Then the result extends by induction. \(\square\)

### 5. PROOFS OF THE MAIN RESULTS

**Proof of Theorem 3.1.** Relation (1.5) is the starting point of the present proof. We consider (3.1) as a conjunction of two asymptotic inequalities:
\[
\psi_r(x) \leq \sum_{k=1}^{\infty} F((1 + r)^k x) \quad \text{and} \quad \psi_r(x) \geq \sum_{k=1}^{\infty} F((1 + r)^k x). \tag{5.1}
\]

Now we aim at the first relation in (5.1). For any $m = 1, 2, \ldots$ we derive
\[
\max_{0 \leq n < \infty} \sum_{k=0}^{n} X_k (1 + r)^{-k} \leq \max_{1 \leq n \leq m} \sum_{k=1}^{n} X_k (1 + r)^{-k} + \sum_{k=m+1}^{\infty} X_k^+ (1 + r)^{-k} = A_m + B_m.
\]
We follow the proofs of Lemma 4.24 of Resnick (1987) and Proposition 1.1 of Davis and Resnick (1988) to show that $P(B_m > x)$ is asymptotically negligible when compared with $P(A_m > x)$ in case $x$ and $m$ are sufficiently large; see also Embrechts et al. (1997, Section A3.3) for a simpler treatment. For all large $m = 1, 2, \ldots$ we have
For this fixed $m$ Combining (5.3) and (5.2) and applying Lemma 4.5 we obtain

$$
\Pr(B_m > x) \leq \Pr\left(\sum_{k=m+1}^{\infty} (1+r)^{-k} X_k^+ > \sum_{k=m+1}^{\infty} (1+r)^{-k}(1+r)^{k/2}x\right)
\leq \sum_{k=m+1}^{\infty} \Pr(X_k^+ > (1+r)^{k/2}x),
$$

Applying inequality (4.1) we obtain that for all large $x > 0$,

$$
\frac{\Pr(B_m > x)}{\Pr(A_m > x)} \leq \sum_{k=m+1}^{\infty} \frac{\Pr(X > (1+r)^{k/2}x)}{\Pr(X > (1+r)x)} \leq C_1 \sum_{k=m+1}^{\infty} (1+r)^{-\alpha(k/2-1)},
$$

Hence, for any $0 < \epsilon < 1$ we can choose some $m = 1, 2, \ldots$ such that for all large $x$,

$$
\Pr(B_m > x) \leq \epsilon \Pr(A_m > x).
$$

(5.2)

For this fixed $m$, by Lemmas 4.6 and 4.7, $A_m$ is subexponentially distributed, satisfying

$$
\Pr(A_m > x) \sim \sum_{k=1}^{m} \Pr(X_k(1+r)^{-k} > x) \leq \sum_{k=1}^{\infty} \Pr(X > x(1+r)^k),
$$

(5.3)

Combining (5.3) and (5.2) and applying Lemma 4.5 we obtain

$$
\psi_r(x) \leq \Pr(A_m + B_m > x) \leq (1+\epsilon) \Pr(X > x(1+r)^k).
$$

Thus the first relation in (5.1) follows from the arbitrariness of $\epsilon > 0$.

Next we start to prove the second relation in (5.1). For any $m = 1, 2, \ldots$, applying Lemmas 4.6 and 4.7 once again we have

$$
\psi_r(x) \geq \Pr\left(\max_{1 \leq n \leq m} \sum_{k=1}^{n} X_k(1+r)^{-k} > x\right) \sim \left(\sum_{k=1}^{\infty} - \sum_{k=m+1}^{\infty}\right) \Pr(X(1+r)^{-k} > x).
$$

Along the same line as above, we can obtain that for large $m = 1, 2, \ldots$,

$$
\sum_{k=m+1}^{\infty} \Pr(X(1+r)^{-k} > x) \leq \epsilon \Pr(X(1+r)^{-1} > x) \leq \epsilon \Pr(X(1+r)^{-k} > x),
$$

Hence,

$$
\psi_r(x) \geq (1-\epsilon) \sum_{k=1}^{\infty} \Pr(X(1+r)^{-k} > x).
$$

The second relation in (5.1) follows from the arbitrariness of $\epsilon > 0$. □

**Proof of Corollary 3.1.** We start the proof from (3.1). Clearly, the dominated convergence theorem, which is ensured by inequality (4.1), allows the interchange of the limit and sum in the following derivation:
\[
\lim_{x \to \infty} \frac{\psi_r(x)}{F(x)} = \lim_{x \to \infty} \sum_{k=1}^{\infty} \frac{F((1 + r)^k x)}{F(x)} = \sum_{k=1}^{\infty} \lim_{x \to \infty} \frac{F((1 + r)^k x)}{F(x)}.
\]

Hence, relation (3.2) follows immediately from the definition of the class \( R \). Relation (3.3) can be proved similarly. \( \square \)

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