A large deviation result for aggregate claims with dependent claim occurrences

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Abstract

In this paper we study precise large deviations of a compound sum of claims, in which the claims arrive in groups and the claim numbers in the groups may follow a certain negative dependence structure. We try to build a platform both for the classical large deviation theory and for the modern stochastic ordering theory.

1. Introduction

Inspired by the recent works of Cline and Hsing (1991), Klüppelberg and Mikosch (1997), Tang et al. (2001), and Ng et al. (2004), in the present paper we are interested in precise large deviations of the random sum

\[ S_t = \sum_{k=1}^{N_t} X_k, \quad t \geq 0. \] (1.1)

Here \( \{X_k, k = 1, 2, \ldots\} \) is a sequence of independent, identically distributed (i.i.d.), and nonnegative heavy-tailed random variables, representing the sizes of successive claims, with common distribution function \( F = 1 - \bar{F} \) and finite mean \( \mu > 0 \). \( \{N_t, t \geq 0\} \) is a counting process (that is, a nonnegative, nondecreasing, and integer-valued process) with \( N_t \) representing the number of claims by time \( t \), with a finite mean function \( \lambda_t = E[N_t] \to \infty \) as \( t \to \infty \); and, as usual, \( \sum_{k=1}^{\infty} (\cdot) = 0 \) by convention. The process \( \{N_t, t \geq 0\} \) and the sequence \( \{X_k, k = 1, 2, \ldots\} \)
are assumed to be mutually independent. Our goal is to establish a precise large deviation result that for each fixed \( \gamma > 0 \), the relation
\[
\Pr(S_t - \mu \lambda t > x) \sim \frac{\lambda_t}{\bar{F}(x)}, \quad t \to \infty,
\]
holds uniformly for all \( x \geq \gamma \lambda t \). Hereafter, all limit relationships are for \( t \to \infty \) unless stated otherwise. The uniformity of relation (1.2) means
\[
\lim_{t \to \infty} \sup_{x \geq \gamma \lambda t} \left| \frac{\Pr(S_t - \mu \lambda t > x)}{\lambda_t \bar{F}(x)} - 1 \right| = 0.
\]
This is crucial for our purpose.

As an application of (1.2), we consider the calculation of the stop-loss premium of the random sum \( S_t \) with large retention \( d \), say \( d = d(t) \geq (\mu + \gamma) \lambda t \). Write \( x = \max\{x, 0\} \). As \( t \) increases, applying the uniformity of the asymptotic relation (1.2) we have
\[
\mathbb{E}[(S_t - d)_+] = \int_0^{\infty} \Pr(S_t > x) \, dx = \int_0^{\infty} \Pr(S_t - \mu \lambda t > x - \mu \lambda t) \, dx \sim \lambda \int_0^{\infty} \bar{F}(x - \mu \lambda t) \, dx = \lambda \mathbb{E}[X_1 + \mu \lambda t - d].
\]
Clearly, the calculation of the right-hand side of the above is much simpler than the calculation of the stop-loss premium of \( S_t \) itself. For further applications of precise large deviations to insurance and finance, we refer the reader to Klüppelberg and Mikosch (1997), Mikosch and Nagaev (1998), and Embrechts et al. (1997, Chapter 8), among many others.

In this paper we shall consider the following special case of the random sum (1.1), in which the claim arrivals follow a compound process described below:

1. the arrival times \( 0 = \sigma_0 < \sigma_1 < \sigma_2 < \cdots \) constitute a counting process
\[
\tau_t = \sum_{k=1}^{\infty} 1_{(\sigma_k \leq t)}, \quad t \geq 0,
\]
with a finite mean function \( \mathbb{E}[\tau_t] \to \infty \), where \( 1_A \) denotes the indicator function of \( A \);
2. at each arrival time \( \sigma_k \), a group of \( Z_k \) claims arrives, and \( Z_k, k = 1, 2, \ldots, \) constitute a sequence of nonnegative, integer-valued, and identically distributed random variables, that may be independent, but can also follow a certain dependence structure;
3. the number of claims by time \( t \) is therefore a process
\[
N_t = \sum_{k=1}^{\infty} Z_k 1_{(\sigma_k \leq t)} = \sum_{k=1}^{\tau_t} Z_k, \quad t \geq 0;
\]
4. all sources of randomness, \( \{X_k, k = 1, 2, \ldots, \} \), \( \{Z_k, k = 1, 2, \ldots, \} \), and \( \{\tau_t, t \geq 0\} \), are mutually independent.

Clearly, if each \( Z_k \) is degenerate at 1, the model above reduces to the ordinary risk model. Generally speaking, however, it describes a nonstandard risk model since the random sum (1.1) is equal to
\[
S_t = \sum_{k=1}^{\tau_t} X_k + \sum_{k=1}^{\tau_{t+}} X_k + \sum_{k \geq \tau_{t+} + Z_{\tau_{t+}+1}} X_k = A_1 + A_2 + \cdots + A_{\tau_t},
\]
where \( \{A_n, n = 1, 2, \ldots\} \), though independent of \( \{\tau_t, t \geq 0\} \), is no longer a sequence of i.i.d. random variables.
A reference related to the present model is Denuit et al. (2002), who interpreted the random variable \( Z_k \) above as the occurrence of the \( k \)th claim, hence as a Bernoulli variate \( I_k, k = 1, 2, \ldots \), and who assumed that the sequence \( \{I_k, k = 1, 2, \ldots \} \) follows a certain dependence structure. In this way, the random sum (1.1) is equal to

\[
S_t = \sum_{k=1}^{t} X_k I_k.
\]

See also Ng et al. (2004, Section 5.1).

The remaining part of the paper is organized as follows. Section 2 recalls some preliminaries about heavy-tailed distributions, Section 3 establishes the precise large deviation result for a standard case where the claim numbers in groups, \( Z_k, k = 1, 2, \ldots \), are independent; and, after introducing a kind of negative dependence structure, Section 4 extends the result to a nonstandard case where the numbers of occurrences \( Z_k, k = 1, 2, \ldots \), follow this dependence structure.

2. Distributions with consistent variation

In this paper, we shall assume that the common claim size distribution \( F \) is heavy tailed. More precisely, we assume that \( F \) has a consistent variation and we write \( F \in C \). By definition, a distribution function \( F \) belongs to the class \( C \) if

\[
\limsup_{y \to 1} \lim_{x \to \infty} \frac{\bar{F}(xy)}{\bar{F}(x)} = 1.
\]

(2.1)

Discussions and applications of this class can be found, for example, in Cline (1994), Schlegel (1998), Jelenković and Lazar (1999, Section 4.3), and Ng et al. (2004), and Tang (2004).

Specifically, the class \( C \) covers the famous class \( R \), which consists of all distribution functions with regularly varying tails in the sense that there is some \( \alpha > 0 \) such that the relation

\[
\lim_{x \to \infty} \frac{\bar{F}(xy)}{\bar{F}(x)} = y^{-\alpha}
\]

(2.2)

holds for each \( y > 0 \). We denote by \( F \in R - \alpha \) the regularity property in (2.2).

A simple example to illustrate that the inclusion \( R \subset C \) is strict is the distribution function of the random variable \( X = (1 + U)^2 \mathbb{N} \), where \( U \) and \( \mathbb{N} \) are independent random variables with \( U \) uniformly distributed on \((0, 1)\) and \( \mathbb{N} \) geometrically distributed satisfying \( \Pr(N = n) = (1 - p)p^n \) for \( 0 < p < 1 \) and \( n = 0, 1, \ldots \); see Kaas et al. (2004), and Cai and Tang (2004).

For a distribution function \( F \), as done recently by Tang and Tsitsiashvili (2003), we define

\[
J^*_F = -\lim_{y \to \infty} \frac{\log F_F(y)}{\log y}
\]

with \( F_F(y) = \liminf_{x \to \infty} \frac{F_F(xy)}{F_F(x)} \) for each \( y > 0 \),

(2.3)

where the existence of the first limit is due to Theorem 2.1.5 of Bingham et al. (1987). We call \( J^*_F \) the upper Matuszewska index of the distribution function \( F \). For more details of the Matuszewska indices, see Bingham et al. (1987, Chapter 2.1), Cline and Samorodnitsky (1994), as well as Tang and Tsitsiashvili (2003). Clearly, if \( F \in C \) then \( J^*_F < \infty \), and if \( F \in R - \alpha \) then \( J^*_F = \alpha \).

Tang and Tsitsiashvili (2003, Lemma 3.5) proved the following result.

**Lemma 2.1.** For a distribution function \( F \) with upper Matuszewska index \( J^*_F < \infty \), it holds for each \( p > J^*_F \) that \( x^{-p} = o(F_F(x)) \).

By this lemma, \( J^*_F \leq 1 \) should hold if \( F_F(x) \) has a finite mean.
3. A standard case with independent occurrences

3.1. A preliminary result

Ng et al. (2004) investigated the random sum (1.1) and obtained the following general result:

**Proposition 3.1.** Consider the random sum

\[ \sum_{k=1}^{N_t} Z_k. \]

(1.1)

If \( F \in \mathcal{C} \) and \( N_t \) satisfies

\[ \mathbb{E}\left[ N_t \mid N_t > \eta \lambda t \right] = O(\lambda t) \]

(3.1)

for some \( p > I_2^* \) and all \( \eta > 1 \), then for each fixed \( \gamma > 0 \), the precise large deviation result (1.2) holds uniformly for all \( x \geq \gamma \lambda t \).

Now we state a preliminary result that is interesting in its own right.

**Theorem 3.1.** Consider the compound model introduced in Section 1. If the counting process \( \tau_t \) satisfies

\[ \mathbb{E}\left[ \tau_t \mid \tau_t > \eta \mathbb{E}\tau_t \right] = O(\mathbb{E}\tau_t) \]

(3.2)

for some \( r > 0 \) and all \( \eta > 1 \), and the claim numbers \( Z_k, k = 1, 2, \ldots, \) are independent with

\[ \mathbb{E}[Z_k] < \infty. \]

(3.3)

then relation (3.1) holds for each \( p \in (0, r) \) and \( \eta > 1 \). Consequently, if further \( F \in \mathcal{C} \) and (3.2) and (3.3) hold for some \( r > I_2^* \), then for each fixed \( \gamma > 0 \), the precise large deviation result (1.2) holds uniformly for all \( x \geq \gamma \lambda_t \).

Actually, condition (3.2) is fulfilled by a lot of commonly used counting processes such as the renewal counting process, the Cox process, and the inhomogeneous Poisson process. In addition, the second part of Theorem 3.1 improves Theorems 2.3 and 2.4 of Tang et al. (2001) in several directions.

3.2. An inequality

Before giving the proof of Theorem 3.1, we show, in the spirit of Fuk and Nagaev (1971) (see also Nagaev (1976) for additional erratum and extension) and Nagaev (1979), a general inequality for the tail probability of sums of i.i.d. random variables.

**Lemma 3.1.** Let \( \{Z_k, k = 1, 2, \ldots, \} \) be a sequence of i.i.d. nonnegative random variables with \( \mathbb{E}[Z_1] < \infty \) for some \( r \geq 1 \). Then for each \( \gamma > \mathbb{E}[Z_1] \), there is some \( C > 0 \) not depending on \( Z \) and \( m \) such that for all \( m = 1, 2, \ldots, \) and \( x \geq y m \),

\[ \mathbb{P}\left( \sum_{k=1}^{m} Z_k > x \right) \leq \frac{C m}{x^r}. \]

(3.4)

**Proof.** For the case \( r = 1 \), (3.4) is a trivial consequence of Chebyshev’s inequality. Now we assume \( r > 1 \) and define \( r = \min\{r, 2\} \). With an arbitrarily fixed constant \( v > 0 \), by Theorem 2 of Fuk and Nagaev (1971) (see also Theorem 12 of Nagaev (1979)), we obtain

\[ \mathbb{P}\left( \sum_{k=1}^{m} Z_k > x \right) \leq m \mathbb{P}(Z_1 > vx) + P_t(x) \]

(3.5)
with 
\[ P_v(x) = \exp \left\{ \frac{1}{v} - \frac{1}{v} - m \frac{E[Z_1 \mathbb{1}_{\{Z_1 \leq vx\}}]}{\tau} \log \left( \frac{E[Z_1 \mathbb{1}_{\{Z_1 \leq vx\}}]}{m} + 1 \right) \right\}. \]

Since \( x \geq \gamma m \) and \( \gamma > E[Z_1] \), some simple calculations lead to 
\[ P_v(x) \leq e^{\frac{1}{v}} \left( \gamma v \tilde{r} - 1 \right) E[Z_1] \left( 1 - \frac{E[Z_1]}{\gamma} \right) / v \]

It follows that for all small \( v \), say \( 0 < v \leq v_0 \), 
\[ P_v(x) = o\left( x^{-r} \right). \]

For the first term on the right-hand side of (3.5), by Chebyshev’s inequality, it holds that 
\[ m \Pr(Z_1 > vx) \leq (vx - rm E[Z_1]). \]

This proves that inequality (3.4) holds for some constant \( C > 0. \)

3.3. Proof of Theorem 3.1

It suffices to prove that under conditions (3.2) and (3.3), relation (3.1) holds for each \( p \in (0, r) \) and \( \eta > 1. \) For this purpose we recall an elementary inequality that for all real numbers \( a_1, a_2, \ldots \) and each \( r \geq 0, \)
\[ \left| \sum_{k=1}^{m} a_k \right|^r \leq \max\{m^{r-1}, 1\} \sum_{k=1}^{m} |a_k|^r. \] (3.6)

By this inequality and relation (1.4), one easily checks that for each \( t > 0 \) and for the number \( r \) given in (3.3), 
\[ E[N_t^p] \leq E[\tau_t^r] E[Z_1^r] < \infty. \] (3.7)

Denote by \( \Delta \) the forward difference operator and by \( \lfloor a \rfloor \) the largest integer that is not larger than \( a. \) In view of (3.7), for each \( p \in (0, r) \) and \( \eta > 1, \) we can use summation by parts to obtain 
\[ E[N_t^p \mathbb{1}_{\{N_t > \eta \}}] = \sum_{n=\lfloor \eta \rfloor}^{\infty} n^p \Pr(N_t = n) = - \sum_{n=\lfloor \eta \rfloor}^{\infty} n^p \Delta \Pr(N_t \geq n) \]
\[ = \sum_{n=\lfloor \eta \rfloor}^{\infty} \Pr(N_t \geq n) \Delta n^p + \left( \lfloor \eta \rfloor + 1 \right) \Pr(N_t > \eta) = I_1(t) + I_2(t). \] (3.8)

Recalling relation (1.4), we deal with \( I_1(t) \) as 
\[ I_1(t) \sim \sum_{n=\lfloor \eta \rfloor}^{\infty} n^{p-1} \Pr(N_t \geq n) \sim \sum_{n=\lfloor \eta \rfloor}^{\infty} n^{p-1} \left( \sum_{\nu = \lfloor \eta \rfloor}^{n} \Pr(\tau_\nu = n) + \sum_{\nu = \lfloor \eta \rfloor}^{n} \Pr(\tau_\nu = n) \right) \]
\[ = I_1(t) + I_2(t). \]
Note that
\[ \lambda_t = E[N_t] = E[Z_t]E[\tau_t]. \]  
(3.9)

By Lemma 3.1, we know that for some \( C_1 > 0 \),
\[ I_{11}(t) \leq C_1 E[Z_t] \sum_{n > \eta\lambda_t} n^{p-1} \Pr(\sum_{k=1}^{n} Z_k \geq n) \leq o(\lambda_t). \]

Successively applying Chebyshev’s inequality and (3.6), we obtain that
\[ I_{12}(t) = \sum_{n > \eta\lambda_t} n^{p-1} \Pr(\sum_{k=1}^{n} Z_k \geq n) \Pr(\tau_t = m) \leq C_2 \lambda_t \sum_{m > \sqrt{\eta E[\tau_t]}} \Pr(\tau_t = m). \]

Hence by condition (3.2), \( I_{12}(t) = o(\lambda_t) \). This proves that
\[ I_1(t) = o(\lambda_t). \]  
(3.10)

Now we turn to \( I_2(t) \). Analogously, for some \( C_2 > 0 \),
\[ I_2(t) \leq C_3 \lambda_t \sum_{m > \sqrt{\eta E[\tau_t]}} \Pr(\tau_t = m) = o(\lambda_t). \]

By condition (3.2), it is also easy to see that
\[ I_{22}(t) \leq C_3 \lambda_t^p \Pr(\tau_t > \sqrt{\eta E[\tau_t]}) = o(\lambda_t). \]

This proves that
\[ I_2(t) = o(\lambda_t). \]  
(3.11)

Plugging (3.10) and (3.11) into (3.8) we finally obtain that
\[ E[N_t]^p 1_{(N_t > \eta\lambda_t)} = o(\lambda_t). \]

Hence, relation (3.1) holds for each \( p \in (0, r) \) and \( q > 1 \). This ends the proof of Theorem 3.1. □
4. A nonstandard case with dependent occurrences

4.1. An equivalent statement of assumption (3.1)

As we have seen in Section 3, the proof of Theorem 3.1 heavily relies on the independence assumptions made there. In the following result, we rewrite the left-hand side of (3.1) as the expectation of a nondecreasing and convex function of $\mathcal{N}_t$. This enables us to check some nonstandard cases by using the well-developed stochastic ordering theory.

Lemma 4.1. Let $\{\mathcal{N}_t, t \geq 0\}$ be a nonnegative process with a finite mean function $\lambda_t = E[\mathcal{N}_t] \to \infty$. Then for each fixed $p > 0$, the following two assertions are equivalent:

A. for all $\eta > 1$,
$$E[N_t^p 1_{\{\mathcal{N}_t > \eta \lambda_t\}}] = O(\lambda_t), \quad (4.1)$$
B. for all $\eta > 1$,
$$E[(\mathcal{N}_t - \eta \lambda_t)^p] = O(\lambda_t). \quad (4.2)$$

Proof. The implication $A \implies B$ is trivial since
$$E[(\mathcal{N}_t - \sqrt{\eta} \lambda_t)^p] \leq E[N_t^p 1_{\{\mathcal{N}_t > \eta \lambda_t\}}].$$
To verify the other implication $B \implies A$, let $\eta > 1$ be arbitrarily fixed. We have
$$E[(\mathcal{N}_t - \sqrt{\eta} \lambda_t)^p] = E((\mathcal{N}_t - \sqrt{\eta} \lambda_t)^p 1_{\{\mathcal{N}_t > \sqrt{\eta} \lambda_t\}} + 1_{\{\mathcal{N}_t > \sqrt{\eta} \lambda_t\}}) \geq E((\mathcal{N}_t - \sqrt{\eta} \lambda_t)^p 1_{\{\mathcal{N}_t > \sqrt{\eta} \lambda_t\}}) \geq (1 - \sqrt{\eta}/\eta)E[N_t^p 1_{\{\mathcal{N}_t > \sqrt{\eta} \lambda_t\}}].$$
Since by condition $B$ the left-hand side of the above is $O(\lambda_t)$, we immediately obtain relation (4.1) with $\eta > 1$ being arbitrarily given. This ends the proof of Lemma 4.1. \(\square\)

4.2. Negative cumulative dependence and convex order

Recently, Denuit et al. (2001) extended the notion of bivariate positive quadrant dependence (PQD) to arbitrary dimension by introducing the notion of positive cumulative dependence (PCD). The analysis there indicates that PCD can well keep the intuitive meaning of PQD.

In a similar fashion, we introduce the notion of negative cumulative dependence (NCD) as follows. Let $\{Z_1, Z_2, \ldots, Z_m\}$ be a sequence of random variables. For $I \subset \{1, 2, \ldots, m\}$, we denote by $S_I$ the sum of the random variables $Z_k$ whose indices are in the set $I$. We say that the family of random variables $\{Z_1, Z_2, \ldots, Z_m\}$ is NCD if for each set $I \subset \{1, 2, \ldots, m\}$ and each number $k \in \{1, 2, \ldots, m\} - I$, the inequality
$$P(S_I > x_1, S_k > x_2) \leq P(S_I > x_1) P(S_k > x_2)$$
holds for all real numbers $x_1$ and $x_2$. We say that an infinite family of random variables $\{Z_k, k = 1, 2, \ldots\}$ is NCD if each of its finite subfamilies is NCD.

Given two random variables $Y_1$ and $Y_2$, we say that $Y_1$ precedes $Y_2$ in the stop-loss order, written as $Y_1 \leq_{sl} Y_2$, if the inequality
$$E[\phi(Y_1)] \leq E[\phi(Y_2)]$$
(4.3)
holds for all nondecreasing and convex functions \( \phi \) for which the expectations exist. It is worth mentioning that

\[ Y_1 \leq_s Y_2 \text{ and } E[Y_1] = E[Y_2] \text{ if and only if inequality (4.3) holds for all convex functions } \phi \text{ for which the expectations exist.} \]

Reviews on the stochastic ordering can be found in Dhaene et al. (2002a,b).

As usual, we write by \( \{ Z_{\perp k} \}_{k=1}^{\infty} \) the independent version of the sequence \( \{ Z_k \}_{k=1}^{\infty} \). That is, the random variables \( Z_{\perp k} \), \( k = 1, 2, \ldots, \) are mutually independent and for each \( k \in \{ 1, 2, \ldots, \} \) the random variables \( Z_{\perp k} \) and \( Z_k \) have the same marginal distribution. We have the following result:

**Lemma 4.2.** Let \( \{ Z_k \}_{k=1}^{\infty} \) be a sequence of NCD random variables and let \( \{ Z_{\perp k} \}_{k=1}^{\infty} \) be its independent version. Then the inequality

\[ \sum_{k=1}^{m} Z_k \leq_s \sum_{k=1}^{m} Z_{\perp k} \]

holds for each \( m = 1, 2, \ldots, \).

**Proof.** The proof for \( m = 2 \) can be given in a similar way as proof of Theorem 2 of Dhaene and Goovaerts (1996). The remainder of the proof can be given by proceeding along the same lines as in the proof of Theorem 3.1 of Denuit et al. (2001), only changing the directions of some inequalities.

### 4.3. The main result and its proof

Now we are ready to establish the main result of this paper.

**Theorem 4.1.** Consider the compound model introduced in Section 1. We assume that the claim numbers \( Z_k \), \( k = 1, 2, \ldots, \), are NCD, the claim size distribution \( F \) belongs to the class \( \mathcal{C} \), and conditions (3.2) and (3.3) hold for some \( r > J^*_F \) and all \( \eta > 1. \) Then for each fixed \( \gamma > 0, \) the precise large deviation result (1.2) holds uniformly for all \( x \geq \gamma \lambda t, \)

\[ E[(N_t - \eta \lambda t)^p] = O(\lambda t) \]

holds for each \( p \in (J_F^*, r) \) and all \( \eta > 1. \) Therefore, applying Lemma 4.1 once again, the relation

\[ E[(N_t - \eta \lambda t)^{p-1}(N_t - \eta \lambda t)] = O(\lambda t) \]

holds for each \( p \in (J_F^*, r) \) and all \( \eta > 1. \)
holds for \((p, \eta)\) in the same region. By (4.4) we conclude that, as desired, the relation
\[
E[N_t - \eta \lambda_t^p] = O(\lambda_t)
\]
holds for \((p, \eta)\) in the same region. This ends the proof of Theorem 4.1. \(\square\)

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