On the ruin probabilities of a bidimensional perturbed risk model

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Abstract

We follow some recent works to study the ruin probabilities of a bidimensional perturbed insurance risk model. For the case of light-tailed claims, using the martingale technique we obtain for the infinite-time ruin probability a Lundberg-type upper bound, which captures certain information of dependence between the two marginal surplus processes. For the case of heavy-tailed claims, we derive for the finite-time ruin probability an explicit asymptotic estimate.

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1. A bidimensional risk model

Following the recent works of Ambagaspitiya (1998), Cossette and Marceau (2000), Yuen et al. (2002, 2006), Chan et al. (2003), Wang and Yuen (2005), and Cai and Li (2005), we consider a bidimensional insurance risk process perturbed by diffusion, in which the bidimensional surplus process $\mathbf{R}(t) = (R_1(t), R_2(t))$ is described as

\[
\begin{pmatrix}
R_1(t) \\
R_2(t)
\end{pmatrix} = \begin{pmatrix}
u_1 \\ u_2
\end{pmatrix} + t \begin{pmatrix}
c_1 \\ c_2
\end{pmatrix} - \sum_{i=1}^{N(t)} \begin{pmatrix}
X_{1i} \\ X_{2i}
\end{pmatrix} + \begin{pmatrix}
\sigma_1 B_1(t) \\ \sigma_2 B_2(t)
\end{pmatrix}, \quad t \geq 0.
\]

(1.1)

Here, $\bar{X}_i = (X_{1i}, X_{2i})^\top$, $i = 1, 2, \ldots$, denote pairs of claims whose common arrival times constitute a counting process $\{N(t), t \geq 0\}$, while $\bar{u} = (u_1, u_2)^\top$ denotes the initial surplus vector and $\bar{c} = (c_1, c_2)^\top$ the premium rate vector. Naturally, all vectors $\bar{X}_i$ for $i = 1, 2, \ldots, \bar{u}$, and $\bar{c}$ consist of only nonnegative components. Furthermore, $\bar{B}(t) = (B_1(t), B_2(t))^\top$ denotes a standard bidimensional Brownian motion with constant correlation coefficient $r \in [-1, 1]$, while $\sigma_1 \geq 0$ and $\sigma_2 \geq 0$ denote the marginal volatility coefficients of $\bar{B}(t)$.

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Perturbed risk models in the unidimensional case have been discussed by many people since the pioneering work of Dufresne and Gerber (1991). When $\sigma_1 = \sigma_2 = 0$, model (1.1) is reduced to

$$R(t) = \bar{u} + t\bar{c} - \sum_{i=1}^{N(t)} \bar{X}_i, \quad t \geq 0. \tag{1.2}$$

See Chan et al. (2003) and references therein for some background of introducing the bidimensional model (1.2). In particular, as explained by them, multidimensional models with common arrival process like $(1.1)$ and $(1.2)$ describe situations where each claim event might produce more than one type of claim. A typical example is motor insurance where an accident could cause claims for vehicle damages and bodily injuries. A similar phenomena exists in natural catastrophe insurance.

Throughout this paper, we use the following assumptions:

- $\{N(t), t \geq 0\}$ is a Poisson process with intensity $\lambda > 0$;
- $\{\bar{X}_i, i = 1, 2, \ldots\}$ is a sequence of independent copies of the random pair $\bar{X} = (X_1, X_2)^\top$;
- the sources of randomness, $\{\bar{X}_i, i = 1, 2, \ldots\}$, $\{N(t), t \geq 0\}$, and $\{\bar{B}(t), t \geq 0\}$, are mutually independent; and
- $\bar{X}$ has a joint distribution function $F(x_1, x_2)$ and marginal distribution functions $F_1(x_1)$, $F_2(x_2)$.

For two vectors $\bar{x} = (x_1, x_2)^\top$ and $\bar{y} = (y_1, y_2)^\top$, we write $\bar{x} \leq \bar{y}$ if $x_1 \leq y_1$ and $x_2 \leq y_2$, write $\bar{x} < \bar{y}$ if $x_1 < y_1$ and $x_2 < y_2$, and we define other inequalities in a similar way.

We define the infinite-time ruin probability of model (1.1) as

$$\psi(\bar{u}) = P(T_{\max} < \infty \mid R(0) = \bar{u}), \tag{1.3}$$

where $T_{\max}$ is the first time when both $R_1(t)$ and $R_2(t)$ go below 0, that is,

$$T_{\max} = \inf\{t > 0 \mid \bar{R}(t) < \bar{0}\} = \inf\{t > 0 \mid \max\{R_1(t), R_2(t)\} < 0\}$$

with $\inf \emptyset = \infty$ by convention. We simply call $T_{\max}$ the ruin time. Similarly, we define the finite-time ruin probability of model (1.1) as

$$\psi(\bar{u}; T) = P(T_{\max} \leq T \mid R(0) = \bar{u}), \quad T > 0. \tag{1.4}$$

Other types of ruin time, such as

$$T_{\min} = \inf\{t > 0 \mid \min\{R_1(t), R_2(t)\} < 0\}$$

and

$$T_{\sum} = \inf\{t > 0 \mid R_1(t) + R_2(t) < 0\},$$

may be introduced, as done in Chan et al. (2003). Based on these types of ruin time, several alternatives for the ruin probabilities may be defined in parallel. However, we remark that $T_{\max}$ really represents a more critical time than $T_{\min}$. Actually, at time $T_{\min}$, the insurance company may be able to survive more easily because probably only one of its subsidiary companies gets ruined. We also remark that if we use $T_{\sum}$ to define the ruin probabilities then all problems will be reduced to those in the unidimensional model.

In this paper, we only consider the ruin probabilities (1.3) and (1.4) based on $T_{\max}$ of the bidimensional perturbed risk model (1.1). We think that the other alternatives based on $T_{\min}$ and $T_{\sum}$ can be dealt with similarly and that extension of this work to multidimensional models is straightforward. Due to the complexity of the problems, we have not gone deep into this study, but we hope and believe that our work will stimulate more people in the field to consider the ruin probabilities of multidimensional risk models.

The rest of this paper consists of three sections. In Section 2, we consider the case of light-tailed claims. Resorting to the martingale technique we derive a Lundberg-type upper bound for the infinite-time ruin probability $\psi(\bar{u})$ without using any restriction on the dependence structure of $\bar{X}$. We further give some numerical studies in Section 3 to illustrate that the obtained upper bound captures the effects of certain information of dependence between the two marginal surplus processes. In Section 4, we consider the case of heavy-tailed claims with $X_1$ and $X_2$ being independent, and we derive an explicit asymptotic estimate for the finite-time ruin probability $\psi(\bar{u}; T)$.
2. An upper bound for the infinite-time ruin probability via martingale

In this and the next sections we restrict ourselves to the case of light-tailed claims. In addition to the assumptions mentioned in Section 1, we further assume that $X$ has a mean vector $\vec{\mu} = (\mu_1, \mu_2)^T$ and that the safety loading condition $\tilde{c} > \lambda \hat{\mu}$ holds. For notational convenience, we introduce

- $m(s_1, s_2) = \mathbb{E}[\exp \{s_1X_1 + s_2X_2\}]$;
- $f(s_1, s_2) = \lambda \hat{m}(s_1, s_2) - \lambda - c_1s_1 - c_2s_2 + \frac{1}{2} [\sigma_1^2 s_1^2 + 2r \sigma_1 \sigma_2 s_1 s_2 + \sigma_2^2 s_2^2]$;
- $g(s_1, s_2) = \lambda \hat{m}(s_1, s_2) - \lambda - c_1 s_1 - c_2 s_2$;
- $s_0^1 = \sup \{s_1 \mid \hat{m}(s_1, 0) < \infty\}$, $s_0^2 = \sup \{s_2 \mid \hat{m}(0, s_2) < \infty\}$; and
- $G^0 = \{(s_1, s_2) \mid s_1 \geq 0, s_2 \geq 0, \hat{m}(s_1, s_2) < \infty\} \setminus (0, 0)$.

Furthermore, $\mathcal{F} = \{\mathcal{F}_t, t \geq 0\}$ always represents the natural filtration of $\{	ilde{R}(t), t \geq 0\}$.

By the Hölder inequality, it is clear that the set $G^0$ is non-empty provided that $s_0^1 > 0$ and $s_0^2 > 0$. We start with some lemmas:

**Lemma 2.1.** Let $s_0^1 > 0$, $s_0^2 > 0$, and $\sup_{(s_1, s_2) \in G^0} f(s_1, s_2) > 0$. The following statements hold true:

(a) The equation $f(s_1, s_2) = 0$ has at least one root in $G^0$;
(b) For given $l \geq 0$, the equation $f(s_1, ls_1) = 0$ has at most one root in $(0, s_0^1)$;
(c) For given $l \geq 0$, if $v > 0$ solves $f(s_1, ls_1) = 0$, then $f(s_1, ls_1) > 0$ for all $s_1 > v$ and $f(s_1, ls_1) < 0$ for all $0 < s_1 < v$.

**Proof.** (a) Let $s_2 = ls_1$ for some given $l \geq 0$. Clearly,

$$\frac{df(s_1, ls_1)}{ds_1} = \lambda \left[ \frac{\partial \hat{m}(s_1, s_2)}{\partial s_1} + l \frac{\partial \hat{m}(s_1, s_2)}{\partial s_2} \right]_{s_2=ls_1} - c_1 - lc_2 + \sigma_1^2 s_1 + 2lr \sigma_1 \sigma_2 s_1 + l^2 \sigma_2^2 s_1,$$

so that

$$\frac{df(s_1, ls_1)}{ds_1} \bigg|_{s_1=0} = -(c_1 - \lambda \mu_1 - l(c_2 - \lambda \mu_2) < 0.$$

This means that the function $f(s_1, ls_1)$ takes smaller values than $f(0, 0)$ when $s_1$ is in a right neighbor of $s_1 = 0$. When $l = \infty$, the equation $s_2 = ls_1$ characterizes the line $s_1 = 0$, and in this case, we can still show that the function $f(0, s_2)$ takes smaller values than $f(0, 0)$ when $s_2$ is in a right neighbor of $s_2 = 0$. Since $f(0, 0) = 0$ and $0 \leq l \leq \infty$ can be arbitrary, we conclude that $f(s_1, s_2) < 0$ holds for all $(s_1, s_2)$ sufficiently close to the origin $(0, 0)$. By this and the condition $\sup_{(s_1, s_2) \in G^0} f(s_1, s_2) > 0$ we prove (a).

(b) For every $s_1 > 0$ and $l \geq 0$, we have

$$\frac{d^2 f(s_1, ls_1)}{ds_1^2} = \lambda \left[ \frac{\partial^2 \hat{m}(s_1, s_2)}{\partial s_1^2} + 2l \frac{\partial^2 \hat{m}(s_1, s_2)}{\partial s_1 \partial s_2} + l^2 \frac{\partial^2 \hat{m}(s_1, s_2)}{\partial s_2^2} \right]_{s_2=ls_1} + \sigma_1^2 + 2lr \sigma_1 \sigma_2 + l^2 \sigma_2^2$$

$$\geq \lambda \mathbb{E}[(X_1 + lX_2)^2] + \sigma_1 - l \sigma_2 > 0.$$

This means that the function $f(s_1, ls_1)$ is convex in $s_1 \in (0, s_0^1)$, hence the equation $f(s_1, ls_1) = 0$ can have at most one root in $(0, s_0^1)$.

(c) The result is obvious from the convexity of the function $f(s_1, ls_1)$ on $(0, s_0^1)$.

We are going to construct a martingale based on the surplus process $\{	ilde{R}(t), t \geq 0\}$. The same as in the study of the classical unidimensional model, such a martingale acts as a key tool in establishing any Lundberg-type upper bound for the ruin probability.

**Lemma 2.2.** Let $s_1$, $s_2$ be real numbers such that $\hat{m}(s_1, s_2) < \infty$. Then the process

$$M(\tilde{R}(t)) = \exp(-s_1 R(t) - s_2 \tilde{R}(t) - f(s_1, s_2)t), \quad t \geq 0,$$

is an $\mathcal{F}$-martingale with

$$\mathbb{E}[M(\tilde{R}(t))] = \exp(-s_1 u_1 - s_2 u_2), \quad t \geq 0.$$ (2.1)
Proof. Since \( \{N(t), t \geq 0\} \) is a homogeneous Poisson process, for all \( t, h \geq 0 \) we have

\[
\begin{align*}
E[&\exp(-s_1(R_1(t+h) - R_1(t)) - s_2(R_2(t+h) - R_2(t)))] \\
&= \exp(-s_1 c_1 h - s_2 c_2 h)\exp(\lambda \hat{m}(s_1, s_2) h - \lambda h)\exp\left\{ \frac{1}{2} \left[ \sigma^2_1 s_1^2 + 2r \sigma_1 \sigma_2 s_1 s_2 + \sigma^2_2 s_2^2 \right] h \right\} \\
&= \exp\{f(s_1, s_2)h\}.
\end{align*}
\]

This gives that

\[
E[M(\hat{R}(t+h)) \mid \mathcal{F}_t] = E[\exp(-s_1 R_1(t+h) - s_2 R_2(t+h) - f(s_1, s_2)(t+h)) \mid \mathcal{F}_t] \\
= \exp(-s_1 R_1(t) - s_2 R_2(t) - f(s_1, s_2)t) \\
= M(\hat{R}(t)).
\]

Hence, \( M(\hat{R}(t)) \) is a martingale with respect to \( \mathcal{F} \).

Identity (2.1) follows from \( M(\hat{R}(0)) = \exp(-s_1 u_1 - s_2 u_2) \). \( \blacksquare \)

The following lemma explains that \( T_{\max} \) and \( M(\hat{R}(t)) \) can be thought of as a stopping time and a martingale, respectively, with respect to a common filtration.

**Lemma 2.3.** Let \( T_{\max} \) and \( M(\hat{R}(t)), t \geq 0 \) be defined as above. Then, we can select a filtration \( \mathcal{F}' = \{\mathcal{F}'_t, t \geq 0\} \) such that \( T_{\max} \) and \( M(\hat{R}(t)), t \geq 0 \) are a stopping time and a martingale, respectively, with respect to \( \mathcal{F}' \).

**Proof.** The facts listed below are standard and can be found in some advanced textbooks on stochastic processes:

- If \((\Omega, \mathcal{H}, P)\) is a probability space, \((\xi_t, \mathcal{H}_t)\) is a martingale, and \(\mathcal{H}_t\) is a complete \(\sigma\)-algebra of \(\mathcal{H}_t\) with respect to \(P\), then \((\xi_t, \mathcal{H}_t)\) is a martingale. Furthermore, if \(\{\xi_t, t \geq 0\}\) is right-continuous and \(\mathcal{H}_{t+} = \bigcap_{s \geq t} \mathcal{H}_s\), then \((\xi_t, \mathcal{H}_{t+})\) is also a martingale.
- If \(\{\xi_t, t \geq 0\}\) is right-continuous and is adapted to \(\{\mathcal{H}_{t+}, t \geq 0\}\), and \(\tau = \inf\{t > 0 : \xi_t \in G\}\) for an open set \(G\) of a Polish space, then \(\tau\) is a stopping time with respect to \(\{\mathcal{H}_{t+}, t \geq 0\}\).

Let \(\{\mathcal{F}_t, t \geq 0\}\) be a complete \(\sigma\)-algebra of \(\{\mathcal{F}_t, t \geq 0\}\) with respect to \(P\) and let \(\mathcal{F}_{t+} = \bigcap_{s \geq t} \mathcal{F}_s\). Since \(M(\hat{R}(t))\) is an \(\mathcal{F}\)-martingale and is right-continuous, it is also a martingale with respect to \(\{\mathcal{F}_{t+}, t \geq 0\}\). On the other hand, by the definition of \(T_{\max}\) and the fact that \(\{\hat{R}(t), t \geq 0\}\) is a càdlàg process, we see that \(T_{\max}\) is an \(\{\mathcal{F}_{t+}, t \geq 0\}\)-stopping time, hence an \(\{\mathcal{F}_{t+}, t \geq 0\}\)-stopping time since \(\mathcal{F}_{t+} \subset \mathcal{F}_{t+}\).

In this way, selecting \(\mathcal{F}' = \{\mathcal{F}_{t+}, t \geq 0\}\) we obtain that \(T_{\max}\) is an \(\mathcal{F}'\)-stopping time and \(M(\hat{R}(t))\) is an \(\mathcal{F}'\)-martingale, respectively. \( \blacksquare \)

The fact described in **Lemma 2.3** above will be used in the proof of our main result below:

**Theorem 2.1.** If \(s_1^0 > 0, s_2^0 > 0\), and \(sup_{(s_1, s_2) \in G} f(s_1, s_2) > 0\), then

\[
\psi(\bar{u}) \leq \inf_{(s_1, s_2) \in \Delta^0} \exp(-s_1 u_1 - s_2 u_2),
\]

where \(\Delta^0 = \{(s_1, s_2) \in G^0 \mid f(s_1, s_2) = 0\}\).

**Proof.** Keep in mind that, as described in **Lemma 2.3**, \(T_{\max}\) and \(M(\hat{R}(t)), t \geq 0\) are a stopping time and a martingale, respectively, with respect to \(\mathcal{F}' = \{\mathcal{F}_{t+}, t \geq 0\}\). For every \((s_1, s_2)\) such that \(\hat{m}(s_1, s_2) < \infty\), by **Lemma 2.2** we have

\[
\begin{align*}
\exp(-s_1 u_1 - s_2 u_2) &= E[M(\hat{R}(t))] \\
&\geq E[M(\hat{R}(t))1_{\{T_{\max} \leq t\}}] \\
&= E[E[M(\hat{R}(t)) \mid \mathcal{F}_{T_{\max}+}]1_{\{T_{\max} \leq t\}}] \\
&= E[M(\hat{R}(T_{\max})) \mid T_{\max} \leq t]P(T_{\max} \leq t),
\end{align*}
\]

where \(1_A\) denotes the indicator function of an event \(A\). Since for all \((s_1, s_2) \in G^0\),

\[
\exp(-s_1 R_1(T_{\max}) - s_2 R_2(T_{\max})) \geq 1,
\]

we have
rearranging (2.3) leads to

$$P(T_{\text{max}} \leq t) \leq \exp[-s_1u_1 - s_2u_2] \sup_{0 \leq h \leq t} \exp\{f(s_1, s_2)h\}. \quad (2.4)$$

We write $\Delta^- = \{(s_1, s_2) \in G^0 \mid f(s_1, s_2) < 0\}$ and $\Delta^+ = \{(s_1, s_2) \in G^0 \mid f(s_1, s_2) > 0\}$. If $(s_1, s_2) \in \Delta^+$, then the right-hand side of (2.4) tends to $\infty$ as $t \to \infty$ and it becomes nonsensical. Hence, we only need to consider $(s_1, s_2) \in \Delta^- \cup \Delta^0$ in (2.4). It follows that

$$P(T_{\text{max}} \leq t) \leq \inf_{(s_1, s_2) \in \Delta^- \cup \Delta^0} \exp[-s_1u_1 - s_2u_2].$$

Note that, by Lemma 2.1(a), the set $\Delta^0$ is non-empty. Applying Lemma 2.1(c), it is easy to see that the infimum in the above inequality can actually be attained on $\Delta^0$. Thus,

$$P(T_{\text{max}} \leq t) \leq \inf_{(s_1, s_2) \in \Delta^0} \exp[-s_1u_1 - s_2u_2].$$

Letting $t \to \infty$ yields inequality (2.2).

3. Discussions on the obtained upper bound

First, we discuss the effect on the obtained upper bound of the correlation coefficient of the perturbation. Given $l \geq 0$, if $v_l > 0$ solves the equation $g(s_1, ls_1) = 0$ and $v_f > 0$ solves the equation $f(s_1, ls_1) = 0$, then

$$f(v_l, lv_l) = \frac{1}{2}[\sigma_1^2 + 2rl\sigma_1\sigma_2 + \sigma_2^2]v_l^2 \geq 0 = f(v_f, lv_f).$$

By Lemma 2.1(c), it holds that $v_f \leq v_l$, hence that

$$\exp[-v_l(u_1 + lu_2)] \leq \exp[-v_f(u_1 + lu_2)].$$

Taking infimum on both sides of the inequality above over the range $l \in [0, \infty)$, we see that the perturbation term always increases the upper bound obtained in Theorem 2.1 regardless of how correlated the two marginal perturbations are.

Suppose $\sigma_1 > 0$ and $\sigma_2 > 0$. We further study the effect of $r$, the correlation coefficient of $B_1(t)$ and $B_2(t)$. Let $r_1 \leq r_2$. Given $l \geq 0$, suppose that $v_i > 0$ solves the equation $f(s_1, ls_1) |_{r=r_i} = 0$, $i = 1, 2$. Since

$$\frac{\partial}{\partial r} f(s_1, s_2) = \sigma_1 \sigma_2 s_2 \geq 0 \quad \text{for all } (s_1, s_2) \in G^0,$$

$$f(v_2, lv_2) |_{r=r_2} = 0 = f(v_1, lv_1) |_{r=r_1} \leq f(v_1, lv_1) |_{r=r_2}.$$

By Lemma 2.1(c), it holds that $v_2 \leq v_1$. Therefore,

$$\exp[-v_1(u_1 + lu_2)] \leq \exp[-v_2(u_1 + lu_2)].$$

Similarly to the above, taking infimum on both sides of the inequality above over the range $l \in [0, \infty)$, we see that the upper bound obtained in Theorem 2.1 is increasing in $r$.

Let us numerically study the conclusions above. For this purpose, we assume that the distribution function of $\tilde{X}$ belongs to the well-known bivariate Farlie–Gumbel–Morgenstern class. The general form of a distribution function in this class is given by

$$F(x_1, x_2) = F_1(x_1)F_2(x_2) \left(1 + \rho F_1(x_1)F_2(x_2)\right), \quad -\infty < x_1, x_2 < \infty,$$

where $F_1(x_1) = 1 - F_1(x_1)$ and $F_2(x_2) = 1 - F_2(x_2)$ are univariate distribution functions identical to the marginal distribution functions of $F(x_1, x_2)$, and $\rho$ is a real number in the interval $[-1, 1]$. We refer the reader to Kotz et al. (2000) for details of this distribution class. In particular, we point out that the joint distribution function (3.1) possesses a very nice copula function, called the Farlie–Gumbel–Morgenstern copula,

$$C(u_1, u_2) = u_1u_2 [1 + \rho (1 - u_1)(1 - u_2)], \quad 0 < u_1, u_2 < 1.$$
Table 1
Roots (=values × 10⁻²) of Eq. (3.2) with varying l and r

<table>
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<th>l</th>
<th>0.01</th>
<th>1</th>
<th>5</th>
<th>100</th>
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<tbody>
<tr>
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Table 2
Roots (=values × 10⁻²) of Eq. (3.3) with varying l

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</tbody>
</table>

Mathematically, a bivariate copula \( C(u_1, u_2) \) is a bivariate distribution function on the unit square \([0, 1]^2\) with uniform marginal distribution functions on \((0, 1)\). For more details on copulas see Nelsen (1999). In the actuarial literature, there is a growing interest for the use of copulas to model risk dependency.

**Example 3.1.** Let \( \widetilde{X} \) follow the Farlie–Gumbel–Morgenstern distribution function given in (3.1), where \( F_1 \) and \( F_2 \) are exponential distribution functions with means \( \mu_1 = \lambda_1^{-1} \) and \( \mu_2 = \lambda_2^{-1} \), respectively. Simple calculation gives that

\[
\hat{m}(s_1, s_2) = \frac{(1 + \rho)\lambda_1\lambda_2}{(\lambda_1 - s_1)(\lambda_2 - s_2)} + \frac{4\rho\lambda_1\lambda_2}{(2\lambda_1 - s_1)(2\lambda_2 - s_2)} - \frac{2\rho\lambda_1\lambda_2}{(\lambda_1 - s_1)(\lambda_2 - s_2)} - \frac{2\rho\lambda_1\lambda_2}{(\lambda_1 - s_1)(2\lambda_2 - s_2)}.
\]

Substitute \( \lambda_1 = 0.1, \lambda_2 = 0.5, \lambda = 100, \rho = 0.4, c_1 = 1200, c_2 = 240, \sigma_1 = 0.1 \) and \( \sigma_2 = 0.1 \) to the equations

\[
f(s_1, s_2) = \lambda \hat{m}(s_1, s_2) - \lambda - c_1 s_1 - c_2 s_2 + \frac{1}{2} [\sigma_1^2 s_1^2 + 2\rho \sigma_1 \sigma_2 s_1 s_2 + \sigma_2^2 s_2^2] = 0 \tag{3.2}
\]

and

\[
g(s_1, s_2) = \lambda \hat{m}(s_1, s_2) - \lambda - c_1 s_1 - c_2 s_2 = 0, \tag{3.3}
\]

but let \( r \) vary. Using the software <Mathematica3.0>, we may numerically solve Eq. (3.2) with various values of the slope \( l \) (\( s_2 = ls_1 \)) and the correlation coefficient \( r \); see Table 1. The results in Table 1 confirm that the stronger/weaker correlation of \( B(t) \) leads to a bigger/smaller upper bound for the ruin probability obtained in Theorem 2.1.

Also, we may numerically solve Eq. (3.3) with various values of the slope \( l \) (\( s_2 = ls_1 \)); see Table 2. Comparing Tables 1 and 2, we see that the perturbation does increase the upper bound regardless of whether or not the correlation coefficient \( r \) is positive. □

Next, we discuss the effect of the volatility coefficients of the perturbation. Let \( r > 0 \). Note that for all \((s_1, s_2) \in G^0\),

\[
\frac{\partial f(s_1, s_2)}{\partial \sigma_1} = r \sigma_2 s_1 s_2 + \sigma_1 s_1^2 \geq 0, \quad \frac{\partial f(s_1, s_2)}{\partial \sigma_2} = r \sigma_1 s_1 s_2 + \sigma_2 s_2^2 \geq 0.
\]

Thus, using similar arguments as before, we may conclude that the upper bound obtained in Theorem 2.1 is increasing in \( \sigma_1 \) and \( \sigma_2 \). However, when \( r < 0 \), a similar conclusion may not be drawn.

The example below gives some numerical results to illustrate the effect of the volatility coefficients of the perturbation.

**Example 3.2.** Let \( \widetilde{X} \) follow the Farlie–Gumbel–Morgenstern distribution function given in (3.1), where \( F_1 \) and \( F_2 \) are exponential distribution functions with means \( \mu_1 = \lambda_1^{-1} \) and \( \mu_2 = \lambda_2^{-1} \), respectively. Let \( \lambda_1 = 0.1, \lambda_2 = 0.5, \lambda = 100, \rho = 0.4, c_1 = 1200, c_2 = 240 \) and \( r = \pm 0.8 \), but let \( \sigma_1 \) and \( \sigma_2 \) vary. We may numerically solve Eq. (3.2) with various values of the slope \( l \) (\( s_2 = ls_1 \)), the parameters \( \sigma_1 \) and \( \sigma_2 \); see Table 3. □
Finally, we discuss the effect of the correlation coefficient of the claims. We assume that $\bar{X}$ follows the bivariate Farlie–Gumbel–Morgenstern distribution function given in (3.1). We point out that the relation

$$\text{Cov}(X_1, X_2) = \rho m_1 m_2$$

holds if

$$m_1 = \int_{-\infty}^{\infty} F_1(x_1) F_1(x_1) dx_1 < \infty, \quad m_2 = \int_{-\infty}^{\infty} F_2(x_2) F_2(x_2) dx_2 < \infty.$$

Hence, $\rho$ may serve as an alternative for the correlation coefficient of $\bar{X}$. Because of this, we only analyze the effect of the parameter $\rho$.

Applying Theorems 1 and 2 of Dhaene and Goovaerts (1996), it is not difficult to check that $\hat{m}(s_1, s_2) |_{\rho}$, hence $f(s_1, s_2) |_{\rho}$, is increasing in $\rho$. Actually, for $-1 \leq \rho_1 \leq \rho_2 \leq 1$, it holds for all $\bar{x}$ that $F(\bar{x}) |_{\rho_1} \leq F(\bar{x}) |_{\rho_2}$. Therefore, by Theorem 1 of Dhaene and Goovaerts (1996), the vector $\bar{X} |_{\rho_1}$ is less correlated than the vector $\bar{X} |_{\rho_2}$ in the sense specified there. Then by Theorem 2 of Dhaene and Goovaerts (1996), the sum $s_1 X_1 + s_2 X_2 |_{\rho_1}$ is smaller than $s_1 X_1 + s_2 X_2 |_{\rho_2}$ in stop-loss order (hence in convex order because they have the same mean). In this way, we prove that $\hat{m}(s_1, s_2) |_{\rho_1} \leq \hat{m}(s_1, s_2) |_{\rho_2}$.

For given $l \geq 0$, assume that $v_i$ solves $f(s_1, l s_1) |_{\rho_i} = 0$, $i = 1, 2$. It holds that

$$f(v_2, l v_2) |_{\rho_1} = 0 = f(v_1, l v_1) |_{\rho_1} \leq f(v_1, l v_1) |_{\rho_2}.$$

By Lemma 2.1(c), we have $v_2 \leq v_1$. Therefore,

$$\exp[-v_1(u_1 + l u_1)] \leq \exp[-v_2(u_1 + l u_2)].$$

Thus, using similar arguments as before, we may conclude that the upper bound obtained in Theorem 2.1 is increasing in the correlation parameter $\rho$.

Example 3.3. Let $F_1$ and $F_2$ be exponential distribution functions with means $\mu_1 = \lambda_1^{-1}$ and $\mu_2 = \lambda_2^{-1}$, respectively. Let $\lambda_1 = 0.1$, $\lambda_2 = 0.5$, $\lambda = 100$, $c_1 = 1200$, $c_2 = 240$, $\sigma_1 = \sigma_2 = 0.1$, $r = 0.8$, but let $\rho$ vary. Using the software <Mathematica3.0>, we may numerically solve Eq. (3.2) with various values of the slope $l$ ($s_2 = l s_1$) and the parameter $\rho$; see Table 4. The results in Table 4 confirm that both $s_1$ and $s_2$ are decreasing in $\rho \in [-1, 1]$. Hence, the stronger/weaker correlation of $\bar{X}$ leads to a bigger/smaller upper bound for the ruin probability obtained in Theorem 2.1. \hfill $\square$
Table 4

Roots (values $\times 10^{-2}$) of Eq. (3.2) with varying $l$ and $\rho$

<table>
<thead>
<tr>
<th></th>
<th>$l$</th>
<th>0.01</th>
<th>1</th>
<th>5</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho = 0.8$</td>
<td>$s_1$</td>
<td>1.6660556244</td>
<td>1.5718774478</td>
<td>1.0614900</td>
<td>0.0824456673</td>
</tr>
<tr>
<td></td>
<td>$s_2$</td>
<td>0.0166605562</td>
<td>1.5718774478</td>
<td>5.3074300</td>
<td>8.2445667308</td>
</tr>
<tr>
<td>$\rho = 0.4$</td>
<td>$s_1$</td>
<td>1.6663580190</td>
<td>1.5956020186</td>
<td>1.0962353997</td>
<td>0.0828042248</td>
</tr>
<tr>
<td></td>
<td>$s_2$</td>
<td>0.0166635801</td>
<td>1.5956020186</td>
<td>5.481176999</td>
<td>8.2804224827</td>
</tr>
<tr>
<td>$\rho = 0$</td>
<td>$s_1$</td>
<td>1.6666605335</td>
<td>1.6201535916</td>
<td>1.1338200978</td>
<td>0.0831662382</td>
</tr>
<tr>
<td></td>
<td>$s_2$</td>
<td>0.0166660535</td>
<td>1.6201535916</td>
<td>5.6691048884</td>
<td>8.3166238200</td>
</tr>
<tr>
<td>$\rho = -0.4$</td>
<td>$s_1$</td>
<td>1.6669631679</td>
<td>1.645815510</td>
<td>1.1746556070</td>
<td>0.0835317632</td>
</tr>
<tr>
<td></td>
<td>$s_2$</td>
<td>0.0166696317</td>
<td>1.645815510</td>
<td>5.873278352</td>
<td>8.3531762334</td>
</tr>
<tr>
<td>$\rho = -0.8$</td>
<td>$s_1$</td>
<td>1.6672659223</td>
<td>1.6719396912</td>
<td>1.2193000</td>
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</tr>
<tr>
<td></td>
<td>$s_2$</td>
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<td>1.6719396912</td>
<td>6.096500</td>
<td>8.3900853399</td>
</tr>
</tbody>
</table>

4. Asymptotics for the finite-time ruin probability

In this section we turn our attention to the finite-time ruin probability for the case of heavy-tailed claims.

A famous class of heavy-tailed distribution functions is the subexponential class. By definition, a distribution function $F$ on $[0, \infty)$ is said to be subexponential, denoted by $F \in \mathcal{S}$, if $\bar{F}(x) > 0$ for all $x \geq 0$ and the relation

$$\bar{F}^{*n}(x) \sim n\bar{F}(x), \quad x \to \infty,$$

holds for some (or, equivalently, for all) $n = 2, 3, \ldots$, where $F^{*n}$ denotes the $n$-fold convolution of $F$ and ~ means that the quotient of both sides tends to 1 according to the indicated limit procedure. An authoritative review of subexponential distribution functions in the context of extreme value theory can be found in Embrechts et al. (1997). From Table 1.2.6 of this reference we see that the class $\mathcal{S}$ essentially contains three types of distribution functions: Pareto-like, lognormal-like, and heavy-tailed Weibull-like distribution functions.

Let $F \in \mathcal{S}$, we list some well-known properties below, which will be used tacitly; for the first three see Lemma 1.3.5 of Embrechts et al. (1997), for the fourth see Lemma 2.5.2 of Rolski et al. (1999), and for the last one see Lemma 4.2 of Tang (2004):

- $F$ is long-tailed in the sense that $\bar{F}(x + y) \sim \bar{F}(x)$ as $x \to \infty$ for all real $y$;
- for every $\varepsilon > 0$, $e^{\varepsilon x} \bar{F}(x) \to \infty$ as $x \to \infty$;
- for every $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that
  $$\bar{F}^{*n}(x) \leq C_\varepsilon (1 + \varepsilon)^n \bar{F}(x)$$
  holds for all $n = 1, 2, \ldots$ and all $x \geq 0$;
- for a distribution function $G$ on $[0, \infty)$ satisfying $\bar{G}(x) = o(\bar{F}(x))$, it holds that
  $$\bar{F} \ast G(x) \sim \bar{F}(x);$$
- for nonnegative independent random variables $X$ and $Y$ with $X$ distributed by $F$, it holds that
  $$\Pr(X - Y > x) \sim \bar{F}(x), \quad x \to \infty.$$

There are few references in the literature focusing on the asymptotic behavior of the finite-time ruin probability even for the unidimensional case. As an example, consider the finite-time ruin probability $\psi(u; T)$, $T > 0$, of the unidimensional compound Poisson model in which the claim size distribution function $F$ is subexponential. Applying the dominated convergence theorem guaranteed by (4.1) and (4.2), it is not difficult to obtain that

$$\psi(u; T) \sim \lambda T \bar{F}(u), \quad u \to \infty,$$

where $u$ is the initial surplus and $\lambda > 0$ is the intensity of the Poisson arrival process. See Kaas and Tang (2003) for a more general result.
Now we derive an asymptotic estimate for the finite-time ruin probability $\psi(\tilde{u}; T)$ defined in (1.4). In addition to the assumptions mentioned in Section 1, we further assume that both the claim vector $\tilde{X}$ and the bidimensional Brownian motion $\tilde{B}(t)$ consist of independent components, but we do not assume the safety loading condition. In the result and its proof below, the limit procedure is always $(u_1, u_2) \to (\infty, \infty)$.

**Theorem 4.1.** Suppose that $F_1$ and $F_2$ belong to the class $S$. Then, for each fixed time $T > 0$, we have

$$
\psi(\tilde{u}; T) \sim \lambda T (\lambda T + 1) \overline{F}_1(u_1) \overline{F}_2(u_2).
$$

Comparing the results in (4.5) and (4.6), we find that the finite-time ruin probability $\psi(\tilde{u}; T)$ of our model is asymptotically larger than the product of the finite-time ruin probabilities of the two corresponding marginal unidimensional models. Heuristically, this is because the two marginal risk processes of our model (1.1) are commonly governed by the same arrival process.

**Proof of Theorem 4.1.** For notational convenience, write

$$
\overline{B}_j(T) = \inf_{0 \leq t \leq T} B_j(t), \quad \overline{B}_j(T) = \sup_{0 \leq t \leq T} B_j(t), \quad j = 1, 2.
$$

For each $x > 0$ and $j = 1, 2$, by the well-known reflection principle we have

$$
P(\overline{B}_j(T) < -x) = P(\overline{B}_j(T) > x) = 2P(B_j(T) > x).
$$

Therefore,

$$
P(\overline{B}_j(T) < -x) = P(\overline{B}_j(T) > x) = o(F(x)).
$$

Clearly,

$$
\psi(\tilde{u}; T) = P(\tilde{R}(t) < \tilde{0} \text{ for some } 0 < t \leq T \mid \tilde{R}(0) = \tilde{u})
$$

$$
= P \left( \sum_{i=1}^{N(T)} \tilde{X}_i - t\tilde{c} - \frac{\sigma_1 B_1(t)}{\sigma_2 B_2(t)} > \tilde{u} \text{ for some } 0 < t \leq T \right).
$$

(4.7)

First we derive an asymptotic upper bound for $\psi(\tilde{u}; T)$. We have

$$
\psi(\tilde{u}; T) \leq P \left( \sum_{i=1}^{N(T)} \tilde{X}_i - \frac{\sigma_1 B_1(T)}{\sigma_2 B_2(T)} > \tilde{u} \right)
$$

$$
= \sum_{n=0}^{\infty} P(N(T) = n) \prod_{j=1}^{2} P \left( \sum_{i=1}^{n} X_{ji} - \sigma_j \overline{B}_j(T) > u_j \right).
$$

(4.8)

For every $\varepsilon > 0$, by inequality (4.2), there exist constants $C_{\varepsilon}^{(1)}$, $C_{\varepsilon}^{(2)} > 0$ such that for all $n = 1, 2, \ldots$,

$$
P \left( \sum_{i=1}^{n} X_{1i} - \sigma_1 B_1(T) > u_1 \right)
$$

$$
= \int_{-u_1}^{0} P \left( \sum_{i=1}^{n} X_{1i} - x > u_1 \right) P(\sigma_1 B_1(T) = dx) + P(\sigma_1 B_1(T) < -u_1)
$$

$$
\leq C_{\varepsilon}^{(1)} (1 + \varepsilon)^n \int_{-u_1}^{0} P(X_1 - x > u_1) P(\sigma_1 B_1(T) = dx) + P(\sigma_1 B_1(T) < -u_1)
$$

$$
\leq C_{\varepsilon}^{(1)} (1 + \varepsilon)^n P(X_1 - \sigma_1 B_1(T) > u_1) + P(\sigma_1 B_1(T) < -u_1)
$$

$$
\leq C_{\varepsilon}^{(1)} C_{\varepsilon}^{(2)} (1 + \varepsilon)^n \overline{F}_1(u_1).
$$

where in the last step we used the facts $P(X_1 - \sigma_1 B_1(T) > u_1) \sim \overline{F}_1(u_1)$ (see relation (4.4)) and $P(\sigma_1 B_1(T) < -u_1) = o(1) \overline{F}_1(u_1)$. For each fixed $n = 1, 2, \ldots$, by (4.1) and (4.3) we further have

$$
P \left( \sum_{i=1}^{n} X_{1i} - \sigma_1 B_1(T) > u_1 \right) \sim n \overline{F}_1(u_1).
The same discussions on $P\left(\sum_{i=1}^{n} X_{2i} - \sigma_{2} B_{2}(T) > u_{2}\right)$ for $n = 1, 2, \ldots$ are still valid. Therefore, using the dominated convergence theorem, the right-hand side of (4.8) is asymptotic to

$$\sum_{n=0}^{\infty} P(N(T) = n) n^2 F_1(u_1) F_2(u_2) = \lambda T (\lambda T + 1) F_1(u_1) F_2(u_2).$$

This proves that

$$\psi(\bar{u}; T) \leq (1 + o(1)) \lambda T (\lambda T + 1) F_1(u_1) F_2(u_2). \quad (4.9)$$

Next, we derive an asymptotic lower bound for the ruin probability $\psi(\bar{u}; T)$. Starting from (4.7) we have

$$\psi(\bar{u}; T) \geq P\left(\sum_{i=1}^{N(T)} \bar{X}_i - T \bar{c} - \left(\frac{\sigma_1 B_1(T)}{\sigma_2 B_2(T)}\right) > \bar{u}\right)$$

$$\quad = \sum_{n=0}^{\infty} P(N(T) = n) \prod_{j=1}^{2} P\left(\sum_{i=1}^{n} X_{ji} - \sigma_j B_j(T) > u_j + c_j T\right). \quad (4.10)$$

Likewise, for every $\varepsilon > 0$, there exists a constant $D_\varepsilon > 0$ such that for all $n = 1, 2, \ldots$,

$$P\left(\sum_{i=1}^{n} X_{1i} - \sigma_1 B_1(T) > u_1 + c_1 T\right) \leq D_\varepsilon (1 + \varepsilon)^n P\left(X_1 - \sigma_1 B_1(T) > u_1 + c_1 T\right)$$

$$\leq D_\varepsilon (1 + \varepsilon)^n F_1(u_1).$$

Furthermore, for each fixed $n = 1, 2, \ldots$,

$$P\left(\sum_{i=1}^{n} X_{1i} - \sigma_1 B_1(T) > u_1 + c_1 T\right) \sim n F_1(u_1).$$

The same discussions on $P\left(\sum_{i=1}^{n} X_{2i} - \sigma_{2} B_{2}(T) > u_{2} + c_2 T\right)$ for $n = 1, 2, \ldots$ are still valid. Therefore, using the dominated convergence theorem, the right-hand side of (4.10) is also asymptotic to

$$\sum_{n=0}^{\infty} P(N(T) = n) n^2 F_1(u_1) F_2(u_2) = \lambda T (\lambda T + 1) F_1(u_1) F_2(u_2).$$

This proves that

$$\psi(\bar{u}; T) \geq (1 + o(1)) \lambda T (\lambda T + 1) F_1(u_1) F_2(u_2). \quad (4.11)$$

By inequalities (4.9) and (4.11) we obtain the announced relation (4.6). □

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