The impact on ruin probabilities of the association structure among financial risks

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Abstract

We consider a discrete-time insurance risk model, in which the financial risks constitute a stationary process with finite dimensional distributions of Farlie–Gumbel–Morgenstern type. We obtain an exact asymptotic formula for the ruin probability, reflecting the impact of this kind of association structure among the financial risks.

Keywords: Asymptotics; Farlie–Gumbel–Morgenstern distribution; Stationary process; Ruin probability

1. Introduction

Following the recent works of Nyrhinen (1999, 2001) and Tang and Tsitsiashvili (2003, 2004), we consider the discrete-time insurance risk model

\[ S_0 = 0, \quad S_n = \sum_{i=1}^{n} X_i \prod_{j=1}^{i} Y_j, \quad n = 1, 2, \ldots, \]  

where \{\(X_1, X_2, \ldots\)\} is a sequence of independent and identically distributed (i.i.d.) random variables with common distribution \(F\) on \((−\infty, \infty)\), \{\(Y_1, Y_2, \ldots\)\} is another sequence of nonnegative random variables with common distribution \(G\) on \([0, \infty)\) satisfying \(G(0) < 1\), and the two sequences are mutually independent. In this model, the random variable \(X_i\) is understood as the total loss during period \(i\) and the random variable \(Y_j\) as the discount factor from time \(j\) to time \(j - 1\). Thus, the sum \(S_n\) represents the aggregated discounted losses by time \(n\) of an insurer in a stochastic economic environment. According to Norberg (1999), we call \(X_1, X_2, \ldots\) the insurance risks and \(Y_1, Y_2, \ldots\) the financial risks. The function

\[ \psi(x) = \Pr \left( \max_{0 \leq n < \infty} S_n > x \right) \] 

defines the probability of ultimate ruin of the insurer whose initial wealth is \(x \geq 0\).

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We shall assume that the loss distribution \( F \) is regularly varying tailed (to the right). A distribution \( F \) is said to be regularly varying tailed with regularity index \( \alpha > 0 \), denoted by \( F \in \mathcal{A}_{-\alpha} \), if \( \bar{F}(x) = 1 - F(x) > 0 \) for all \( x \)
\[
\bar{F}(xy) \sim y^{-\alpha} \bar{F}(x) \quad \text{for all} \ y > 0.
\]

Hereafter, all limit relationships are for \( x \to \infty \) unless stated otherwise, and, for two positive functions \( a(\cdot) \) and \( b(\cdot) \), we write \( a(x) \sim b(x) \) if \( \lim a(x)/b(x) = 1 \). Denote by \( \mathcal{A} \) the union of all \( \mathcal{A}_{-\alpha} \) over the range \( 0 < \alpha < \infty \).

The class \( \mathcal{A} \) contains the popular Pareto, Burr, and Loggamma distributions. We refer the reader to Bingham et al. (1987) and Embrechts et al. (1997) for more details of this class.

Define \( \mu_t = \int_0^\infty y^t G(dy) \), which is a convex function over some \( t \)-region for which \( \mu_t < \infty \). Tang and Tsitsiashvili (2004) studied the asymptotic behavior of the ruin probability \( \psi(x) \) in a special case with i.i.d. financial risks, and they proved that if \( F \in \mathcal{A}_{-\alpha} \) for some \( \alpha > 0 \) and \( \mu_{x-\delta} < 1 \) for some \( \delta > 0 \), then
\[
\psi(x) \sim \frac{\mu_x}{1 - \mu_x} \bar{F}(x).
\]

In this note we aim to extend the study to a certain nonstandard case with associated financial risks. We use multivariate Farlie–Gumbel–Morgenstern (FGM) distributions to model the \( n \)-dimensional distributions of the financial risks \( \{Y_1, Y_2, \ldots\} \). The result we obtain reflects how seriously the association structure among the financial risks impacts the asymptotic behavior of the ruin probability \( \psi(x) \).

2. The main result

Let us first prepare some preliminaries before stating our main result.

An \( n \)-dimensional FGM distribution has the form
\[
G_{Y_1, \ldots, Y_n}(y_1, \ldots, y_n) = \left( \prod_{i=1}^n G_i(y_i) \right) \left( 1 + \sum_{1 \leq j < k \leq n} a_{jk} \bar{G}_j(y_j \bar{G}_k(y_k)) \right),
\]
where \( G_1, G_2, \ldots \) are the corresponding marginal distributions and \( a_{jk} \) are real numbers. We refer the reader to Kotz et al. (2000) for a general account on multivariate FGM distributions.

In the sequel, since we only consider FGM distributions with an identical marginal distribution \( G \) on \([0, \infty)\), we drop the subscripts from \( G_i, G_j, \) and \( G_k \) in (4). In order for \( G_{Y_1, \ldots, Y_n} \) to be a proper multivariate distribution, the coefficients \( a_{jk} \) have to fulfill the constraint that
\[
1 + \sum_{1 \leq j < k \leq n} e_j \bar{e}_k a_{jk} \geq 0 \quad \text{for all} \ e_j, \bar{e}_k = -M \text{ or } 1 - m,
\]
where \( M \) and \( m \) are the supremum and the infimum of the set \( \{G(x) : -\infty < x < \infty\} \setminus \{0, 1\} \). If \( G \) is continuous, we have \( M = 1 \) and \( m = 0 \). Clearly, if the coefficients \( a_{jk} \) are all zero, then \( \{Y_1, Y_2, \ldots\} \) is an i.i.d. sequence.

Write \( c = \int_0^\infty G(y) \bar{G}(y) dy \), \( \mu_t = \int_0^\infty y^t G(dy) \), and \( r_t = \int_0^\infty y^t G(y) dy \). It is clear that
\[
\text{cov}(Y_j, Y_k) = c^2 a_{jk} \quad \text{for all } 1 \leq j < k \leq n;
\]
see also Cambanis (1977). It is also easy to verify that the expectation \( \text{E}(\prod_{i=1}^n Y_i^t) \) for \( t > 0 \), if it exists, satisfies
\[
\text{E} \left( \prod_{i=1}^n Y_i \right)^t = \mu_{1t}^n + (\mu_{1t} - 2r_t) \mu_{1t}^{n-2} \sum_{1 \leq j < k \leq n} a_{jk}.
\]
From these relations, we see that the general form (4) of FGM distributions allows a wide range of (positive or negative) association structure among the random variables involved, and it also explicitly exhibits this association structure. We shall use this distribution family to model the finite dimensional distributions of the financial risks.

We say that a sequence \( \{Y_1, Y_2, \ldots\} \) is weakly stationary if both the mean function \( \mu = \text{E} Y_j \) and the auto-covariance function \( \gamma(h) = \text{cov}(Y_j, Y_{j+h}) \) exist and do not depend on the subscript \( j = 1, 2, \ldots \).

A basic
Theorem 1. If $F$ where property of the function et al. (2005) obtained the following:

Lemma 1. Consider sequences are mutually independent. Based on the work of Resnick and Willekens (1991), recently Goovaerts $f$ with identical marginal distribution $G$ on $[0, \infty)$. In this case, there is some function $\eta(\cdot) \colon \{1, 2, \ldots \} \to (-\infty, \infty)$ such that

$$\eta(k-j) = a_{jk} = c^{-2}\text{cov}(Y_j, Y_k) \quad \text{for all } 1 \leq j < k.$$ (7)

In this study, the function $\eta(\cdot)$ plays a role similar to the auto-correlation function of the sequence $\{Y_1, Y_2, \ldots \}$. In terms of the function $\eta(\cdot)$, the constraint given in (5) can be restated as

$$1 + \sum_{1 \leq j < k \leq n} e_j e_k \eta(k-j) \geq 0 \quad \text{for all } e_j, e_k = -M \text{ or } 1 - m \text{ and all } n = 2, 3, \ldots.$$ (8)

Let us go back to the risk model introduced in Section 1. As done above, we assume that the financial risks $Y_1, Y_2, \ldots$ constitute a stationary sequence with finite dimensional distributions of FGM type given by (4) and with identical marginal distribution $G$ on $[0, \infty)$. Our main result is the following:

**Theorem 1.** If $F \in \mathcal{R}_\alpha$ for some $\alpha > 0$ and $\mu_{z+\delta} < 1$ for some $\delta > 0$, then

$$\psi(x) \sim \mathcal{F}(x) \left( \frac{\mu_x}{1 - \mu_x} + \frac{(\mu_x - 2\alpha)^2}{\mu_x(1 - \mu_x)} \sum_{i=1}^{\infty} \mu_i^2 \eta(h) \right).$$ (9)

By the condition $\mu_{z+\delta} < 1$ and the convexity in $t$ of the function $\mu_t$, we see that $\mu_t < 1$ holds for all $t \in (0, \alpha + \delta]$. Furthermore, the absolute values of $\eta(\cdot)$ are bounded by $c^{-2}\gamma(0)$. Hence, the series on the right-hand side of (9) converges.

Compared with (3), relation (9) contains one more term, which exactly captures the impact of the association structure of the stationary sequence $\{Y_1, Y_2, \ldots \}$.

3. **Proof**

The proof of Theorem 1 is transparent if we use a lemma stated below. Consider the randomly weighted sum

$$S_n(\theta) = \sum_{i=1}^{n} \theta_i X_i, \quad n = 1, 2, \ldots,$$ (10)

where $\{X_1, X_2, \ldots \}$ is a sequence of i.i.d. random variables with common distribution $F$ on $(-\infty, \infty)$, $\{\theta_1, \theta_2, \ldots \}$ is another sequence of nonnegative and nondegenerate-at-zero random variables, and the two sequences are mutually independent. Based on the work of Resnick and Willekens (1991), recently Goovaerts et al. (2005) obtained the following:

**Lemma 1.** Consider (10) with $F \in \mathcal{R}_\alpha$ for some $\alpha > 0$. We have

$$\Pr \left( \max_{1 \leq n < \infty} S_n(\theta) > x \right) \sim \mathcal{F}(x) \sum_{i=1}^{\infty} E\theta_i^x$$ (11)

if one of the following assumptions holds:

1. $0 < \alpha < 1$ and $\sum_{i=1}^{\infty} E\theta_i^{\alpha+\delta} < \infty$ and $\sum_{i=1}^{\infty} E\theta_i^{\alpha-\delta} < \infty$ for some $0 < \delta < \alpha$;
2. $\alpha \geq 1$ and $\sum_{i=1}^{\infty} (E\theta_i^{\alpha+\delta})^{1/(\alpha+\delta)} < \infty$ and $\sum_{i=1}^{\infty} (E\theta_i^{\alpha-\delta})^{1/(\alpha-\delta)} < \infty$ for some $0 < \delta < \alpha$. 


Proof of Theorem 1. Write $\theta_i = \prod_{j=1}^{i} Y_j$ for $i = 1, 2, \ldots$. From (1) and (2), it follows that

$$\psi(x) = \Pr \left( \max_{1 \leq n < \infty} S_n(\theta) > x \right),$$

where $S_n(\theta)$ has a form given in (10). We use Lemma 1 to complete the proof. Since $\mu_t < 1$ holds for all $t \in (0, x + \delta]$, by relation (6) the condition of Lemma 1 holds. It also follows that

$$\sum_{i=1}^{\infty} \mathbb{E} \theta_i^2 = \frac{\mu_x}{1 - \mu_x} + (\mu_x - 2r_x)^2 \sum_{i=2}^{\infty} \mu_x^{i-2} \sum_{1 \leq j < k \leq i} \eta(k-j). \quad (12)$$

Clearly,

$$\sum_{1 \leq j < k \leq i} \eta(k-j) = \sum_{h=1}^{j} (i-h) \eta(h).$$

Substituting this into the double sum in (12) and exchanging the order of summation, we find that

$$\sum_{i=2}^{\infty} \mu_x^{i-2} \sum_{1 \leq j < k \leq i} \eta(k-j) = \frac{1}{\mu_x(1 - \mu_x)^2} \sum_{h=1}^{\infty} \mu_x^h \eta(h).$$

Therefore,

$$\sum_{i=1}^{\infty} \mathbb{E} \theta_i^2 = \frac{\mu_x}{1 - \mu_x} + (\mu_x - 2r_x)^2 \sum_{h=1}^{\infty} \mu_x^h \eta(h).$$

Hence, relation (9) follows from (11). $\square$

References


