Characterization of upper comonotonicity via tail convex order

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\section{Introduction}

In the study of the riskiness of an insurance portfolio, we are often interested in the total loss as a sum of risk variables. While traditional approaches have been based on the independence assumption among risks, recent trends allow dependent structures so that we can model the real situation more plausibly. Complications due to dependence among risks can be avoided by, or appropriately dealt with through, replacing the original sum by a less attractive one with a simpler dependence structure. Several concepts have been introduced to compare risks in the actuarial literature (see e.g. Denuit et al., 2005; Dhaene et al., 2006). One common way is to consider convex order which uses stop-loss premiums with the same mean.

It is well known that if a random vector with given marginal distributions is comonotonic, then it has the largest sum with respect to convex order (see e.g. Dhaene et al., 2002a). A proof based on geometric interpretation of the comonotonic support can be found in Kaas et al. (2002). Conversely, it is also true that if a sum is maximal in convex order in the set of all random vectors with predetermined marginal distributions, then the underlying random variables must be comonotonic. Cheung proved the claim for the case of continuous marginal distributions in Cheung (2008). Later it was generalized to the case of integrable distributions in Cheung (2010). In Mao and Hu (2011) the authors showed that the problem boils down to a bivariate case and the joint distribution is indeed the minimum of the two marginal distributions. The well-known fact that a random vector is comonotonic if and only if it is pairwise comonotonic is used. It is, however, no longer valid for an upper comonotonic random vector and a counter-example is given in the Appendix of this paper.

The concept of upper comonotonicity was introduced and investigated in Cheung (2009). A characterization using joint distributions and the additivity of value at risk, tail value at risk, and expected shortfall were given. In Dong et al. (2010), the authors proved the additivity of $\alpha$-mixed inverse distribution functions and stop-loss premiums for the sum of upper comonotonic random variables. In Cheung (2010), it was shown that the additivity of value at risk for the confidence level being sufficiently large implies the upper comonotonicity.

On the other hand, as in the comonotonic case, we can consider an ordering between the sum $S$ of the components of a random vector $X$ and the corresponding sum $S'_{\text{uc}}$ of an upper comonotonic...
random vector $X^\text{uc}$. Indeed, if $X^\text{uc}$ coincides with $X$ in the lower tail in distribution and they have the same marginal distributions, then the sum $S$ precedes $S^\text{uc}$ in convex order (for details, see Dong et al., 2010). If there is no coincidence in the lower tail, then the convex order is no longer valid in general and it should be relaxed to describe the upper tail only.

The concept of tail convex order was introduced by Cheung and Vanduffel (forthcoming), who proved that, upon suitable conditions, the sum of the components of an upper comonotonic random vector is the largest in tail convex order. This suggests that upper comonotonicity be related to tail convex order rather than convex order. Therefore, the following characterization question arises: If the sum of the components of a random vector is maximal in tail convex order, then is this random vector upper comonotonic? In the present paper, we shall focus on this question and pursue an answer.

The rest of this paper is organized as follows: Section 2 prepares some necessary definitions and states our main result, Section 3 formulates the proof of the result, Section 4 shows an application to the computation of Haezendonck risk measures, and, finally, Appendix contains some examples to further clarify some fallacies.

2. Main result

The underlying probability space is denoted by $(\Omega, \mathcal{F}, P)$, where real valued random variables or risks $X_i$ and $Y_i$ are defined for $i = 1, \ldots, n$. A subset $C \subseteq \mathbb{R}^n$ is said to be comonotonic if for any $x$ and $y$ in $C$, either $x \leq y$ or $y \leq x$ holds. We call a random vector $X = (X_1, \ldots, X_n)$ to be comonotonic if it has a comonotonic support.

As a generalization of comonotonicity, we consider upper comonotonicity. For any given $d = (d_1, \ldots, d_n) \in [\mathbb{R} \cup \{\infty\}]^n$, the upper quadrant $(d_1, \infty) \times \cdots \times (d_n, \infty)$ is denoted by $U(d)$, the lower quadrant $(-\infty, d_1] \times \cdots \times (-\infty, d_n]$ by $L(d)$, and the remaining complement $\mathbb{R}^n \setminus (U(d) \cup L(d))$ by $R(d)$.

**Definition 2.1** (Cheung, 2009). A random vector $X = (X_1, \ldots, X_n)$ is said to be upper comonotonic if there exist some $d \in [\mathbb{R} \cup \{\infty\}]^n$ and some null set $N$ such that

(i) $[X(\omega) : \omega \in (\Omega \setminus N) \cap U(d)]$ is comonotonic,

(ii) $P(X \in U(d)) > 0$,

(iii) $[X(\omega) : \omega \in (\Omega \setminus N) \cap R(d)]$ is empty.

The usual inverse distribution functions $F_X^{-1}(p)$, $F_X^{+1}(p)$ of a random variable $X : \Omega \rightarrow \mathbb{R}$ are defined by

$F_X^{-1}(p) = \inf\{x \in \mathbb{R} : F_X(x) \geq p\}$,

$F_X^{+1}(p) = \sup\{x \in \mathbb{R} : F_X(x) \leq p\}$,

respectively, with the convention $\inf\emptyset = \infty$ and $\sup\emptyset = -\infty$.

To pick up any point in the closed interval $[F_X^{-1}(p), F_X^{+1}(p)]$, we introduce the so-called $\alpha$-mixed inverse distribution functions of $X$ as

$F_X^{-1(\alpha)}(p) = \alpha F_X^{-1}(p) + (1 - \alpha) F_X^{+1}(p)$, \hspace{1cm} p \in (0, 1).

See e.g. Dhaene et al. (2002a) for related discussions. From now on, the distribution function $F_X$ or $F_Y$ will be denoted by $F$ as long as no confusion arises.

A random variable $X$ is said to precede another random variable $Y$ in stop-loss order (written as $X \preceq_{\text{st}} Y$ if $X$ has less stop-loss premiums than $Y$, i.e. $E[(X - d)_+] \leq E[(Y - d)_+]$ for all $d \in \mathbb{R}$. It is well known that stop-loss order preserves the ordering of TVaR and vice versa, i.e. $X \preceq_{\text{st}} Y$ if and only if TVaR$_p[X] \leq$ TVaR$_p[Y]$ for all $p \in (0, 1)$ (see Dhaene et al., 2006).

A random variable $X$ is said to precede $Y$ in convex order (written as $X \preceq_{\text{co}} Y$ if $E[X] = E[Y]$ and $X \preceq_{\text{st}} Y$. For an overview of recent progresses on comonotonicity-based convex bounds, see Dhaene et al. (2002b), Deelstra et al. (2011), and references therein. In particular, by limiting the range of $d$ in stop-loss order, we are led to the following definition.

**Definition 2.2** (Cheung and Vanduffel, forthcoming). A random variable $X$ is said to precede another random variable $Y$ in tail convex order (written as $X \preceq_{\text{tc}} Y$ if there exists some $d^*$ such that

(i) $P(X \geq d^*) > 0$,

(ii) $\mathbb{E}[X] = \mathbb{E}[Y]$,

(iii) $\mathbb{E}[(X - d)_+] \leq \mathbb{E}[(Y - d)_+]$ for all $d \geq d^*$.

For any given $Y$, the existence of an upper comonotonic random vector $Y^\text{uc}$ satisfying $S \preceq_{\text{tc}} S^\text{uc}$ is shown in Section 4 of Dong et al. (2010), where (here and throughout) $S$ and $S^\text{uc}$ are the sums of the components of $Y$ and $Y^\text{uc}$, respectively.

For any given distribution functions $F_1, \ldots, F_n$, the Fréchet space $\mathcal{R} = \mathcal{R} (F_1, \ldots, F_n)$ is defined to be the set of all random vectors with marginal distributions $F_1, \ldots, F_n$. Note that if we restrict ourselves on $\mathcal{R}$, then stop-loss order is exactly the same as convex order.

Now we are ready to state the main result. In what follows, for any given $A \subseteq \mathbb{R}^n$, its closure, interior, and boundary are denoted by $\overline{A}$, int$(A)$, and bd$(A)$, respectively.

**Theorem 2.3.** For any given marginal distributions $F_1, \ldots, F_n$, if there exists $Y = (Y_1, \ldots, Y_n) \in \mathcal{R} = \mathcal{R} (F_1, \ldots, F_n)$ such that

$X_1 + \cdots + X_n \preceq_{\text{st}} Y_1 + \cdots + Y_n$, \hspace{1cm} \forall (X_1, \ldots, X_n) \in \mathcal{R},

then there exist some $d^* \in \mathbb{R}^n$ and a null set $N \subset \Omega$ such that

(i) $P(Y \in U(d^*)) > 0$,

(ii) $\{Y : Y \in (\Omega \setminus N) \cap U(d^*)\}$ is comonotonic,

(iii) $\{Y : Y \in (\Omega \setminus N) \cap \text{int}(R(d^*))\} = \emptyset$.

Furthermore, if $P(Y \in \text{bd}(U(d^*))) = 0$, then $(Y_1, \ldots, Y_n)$ is upper comonotonic with a threshold in $L(d^*)$.

Note that (i)–(iii) of the theorem are similar to, but not exactly the same as, the three conditions in Definition 2.1. Example A.2 below shows the necessity of the condition $P(Y \in \text{bd}(U(d^*))) = 0$ for $Y$ to be upper comonotonic.

3. Proof of the theorem

The proof of Theorem 2.3 will be given by observing relations between $Y$ and its comonotonic counterpart $Y^\text{uc}$ in terms of stop-loss premiums.

3.1. Lemmas

To begin with, we recall the following lemma:

**Lemma 3.1** (Dhaene et al., 2002a). The stop-loss premiums of the sum $S$ of the components of a comonotonic random vector $Y^\text{uc} = (Y_1^\text{uc}, \ldots, Y_n^\text{uc})$ are given by

$\mathbb{E}[(S - d)_+] = \sum_{i=1}^{n} \mathbb{E}[(Y_i^\text{uc} - d_i)_+]$ for $F_{S^\text{uc}}^{+1}(0) < d < F_{S^\text{uc}}^{-1}(1),

where $d_i = F_i^{-1(\alpha)}(F_i^+(d))$ and $\alpha$ solves the equation $F_i^{-1(\alpha)}(F_i^+(d)) = d$.

Note that a similar result also holds true for an upper comonotonic random vector (see e.g. Dong et al., 2010). In addition, we can see that $d_i = F_i^{-1(\alpha)}(q)$ is defined only for values of the form $q = F_i(d)$. Nevertheless, the applicability of Lemma 3.1 during the proof of Theorem 2.3 is guaranteed by the following lemma, which shows that the image of each $F_i$ is a subset of the image of $F_1^+$, namely, $\text{Im}(F_i) \subseteq \text{Im}(F_1^+)$. Indeed, $\text{Im}(F_i) \subseteq \text{Im}(F_1^+)$, $i = 1, \ldots, n.$

**Lemma 3.2.** For any random vector $Y = (Y_1, \ldots, Y_n)$ and its comonotonic counterpart $Y^\text{uc}$, we have

$\text{Im}(F_i) \subseteq \text{Im}(F_1^+)$, \hspace{1cm} i = 1, \ldots, n.
Proof. We prove the contraposition. It is well known that the additivity of the $\alpha$-mixed inverse function holds for a comonotonic random vector (Dhaene et al., 2002a):

$$F_{S^\alpha}(q) = \sum_{i=1}^{n} F_i^{-1}(q_i), \quad 0 < q < 1, 0 \leq \alpha \leq 1.$$  \hfill(2)

If $q \not\in \text{Im}(F_S)$, then there exists a sufficiently small $\epsilon > 0$ such that $F_{S^\alpha}^{-1}(q)$ is constant on the interval $[q, q + \epsilon)$. By (2) each $F_i^{-1}(\cdot)$ is also constant on the interval $[q, q + \epsilon)$ since it is non-decreasing. Then, $q \not\in \text{Im}(F_i)$ for each $i \in \{1, \ldots, n\}$ and this completes the proof. \hfill\square

3.2. Proof of Theorem 2.3

The argument of Mao and Hu (2011) motivates the following proof. Since the comonotonic counterpart $\mathcal{Y}$ is also in the Fréchet space $\mathcal{R}$, the condition (1) implies

$$V_1^\alpha + \cdots + V_n^\alpha \leq \alpha Y_1 + \cdots + Y_n.$$  \hfill(3)

By definition there exists $d^\alpha$ such that $P(S \geq d^\alpha) > 0$ and

$$E[(Y_1 + \cdots + Y_n - d_\alpha)] = E[(Y_1^\alpha + \cdots + Y_n^\alpha - d^\alpha)].$$  \hfill(4)

Since $(S - d_\alpha) + \leq \sum_{i=1}^{n} (Y_i - d_i)_\alpha$, we have $(S - d_\alpha) + = \sum_{i=1}^{n} (Y_i - d_i)_\alpha$ almost surely, or, equivalently, there exists a null set $N \subset \Omega$ such that

$$\left( Y_{i}(\omega) - d_i \right) \left( Y_{j}(\omega) - d_j \right) \geq 0, \quad \forall \omega \in \Omega \setminus N, \forall i, j = 1, \ldots, n. \hfill(5)$$

Since $P(S \geq d^\alpha) > 0$, (4) implies

$$P(Y \in U(d^\alpha)) > 0. \hfill(6)$$

We formulate the remaining proof of Theorem 2.3 into the following three steps.

Step 1. For any $(y_i, \gamma_i) \in U(d^\alpha_i, d^\alpha_{i-1})$, the closure of the upper half of $U(d^\alpha_i, d^\alpha_{i-1})$, we have

$$P(Y_i \leq y_i, Y_j \leq y_j) = \min_{k=i,j} P(Y_k \leq y_k).$$

Proof. Note that a distribution function has no more than countably many discontinuities. If we can show that for any $(y_i, \gamma_i) \in U(d^\alpha_i, d^\alpha_{i-1})$ with $y_i$ and $\gamma_i$ being continuities of $F_i$ and $F_j$, there exists a null set $N \subset \Omega$ such that

$$\{\omega \in \Omega \setminus N : Y_i \leq y_i, Y_j \leq y_j\} = \{\omega \in \Omega \setminus N : Y_k \leq y_k\}, \quad k = i \text{ or } j,$$

then the right continuity of distribution functions gives the desired result of Step 1.

Without loss of generality, we may assume that $i = 1, j = 2$, that $F_1, F_2$ are continuous at $y_1, y_2$, and that $q_1 = F_1(y_1) \leq q_2 = F_2(y_2)$. It suffices to prove that

$$P(Y_1 \leq y_1, Y_2 \leq y_2) = P(Y_1 \leq y_1). \hfill(7)$$

Lemma 3.2 guarantees that $q_1, q_2 \in \text{Im}(F_S)$ and we can use (4) to obtain

$$\left( Y_1(\omega) - F_1^{-1}(q_1) \right) \left( Y_2(\omega) - F_2^{-1}(q_2) \right) \geq 0, \quad \forall \omega \in \Omega \setminus N,$$

and

$$\left( Y_1(\omega) - F_1^{-1}(q_2) \right) \left( Y_2(\omega) - F_2^{-1}(q_1) \right) \geq 0, \quad \forall \omega \in \Omega \setminus N,$$  \hfill(8)

where $q_1 \leq q_2$ solve the equations $y_1 = F_1^{-1}(q_1)$ and $y_2 = F_2^{-1}(q_2)$, respectively.

If $q_1 = F_1(y_1) < q_2 = F_2(y_2)$, then, by the continuity of $F_i$ at $y_1$ and (7),

$$\{\omega \in \Omega \setminus N : Y_1 \leq y_1\} \supseteq \{\omega \in \Omega \setminus N : Y_1 < y_1, Y_2 \leq y_2\} \supseteq \{\omega \in \Omega \setminus N : Y_1 \leq y_1, Y_2 \leq F_2^{-1}(q_1)\},$$

and

$$\{\omega \in \Omega \setminus N : Y_2 \leq y_2\} \supseteq \{\omega \in \Omega \setminus N : Y_1 \leq y_1, Y_2 \leq y_2\} \supseteq \{\omega \in \Omega \setminus N : Y_1 \leq y_1, Y_2 \leq F_1^{-1}(q_2)\} \cup N_1,$$

and

$$\{\omega \in \Omega \setminus N : Y_1 \leq y_1\} \supseteq \{\omega \in \Omega \setminus N : Y_1 < y_1, Y_2 \leq y_2\} \supseteq \{\omega \in \Omega \setminus N : Y_1 < y_1, Y_2 \leq F_2^{-1}(q_2)\} \cup N_2,$$

where $P(N_1) = P(N_2) = 0$. By (9) and (10) we conclude that the relation (6) holds, and this completes the proof of Step 1. \hfill\square

Step 2. For any $y = (y_1, \ldots, y_n) \in U(d^\alpha)$,

$$P(Y_1 \leq y_1, \ldots, Y_n \leq y_n) = \min_{i=1,\ldots,n} P(Y_i \leq y_i),$$

and $\mathcal{Y}$ is comonotonic in $U(d^\alpha)$.

Proof. By induction, we will prove the claim that

$$\{\omega \in \Omega \setminus N : Y_i \leq y_i, \ldots, Y_n \leq y_n\} = \{\omega \in \Omega \setminus N : Y_i \leq y_i\} \cap \{\omega \in \Omega \setminus N : Y_k \leq y_k\} \cap \cdots \cap \{\omega \in \Omega \setminus N : Y_{k+1} \leq y_{k+1}\}, \quad k = 1, \ldots, n - 1,$$

for some $i \in \{1, \ldots, n\}$ with null sets $N_i$ and $\tilde{N}$. For $n = 2$, the claim is true by Step 1. Now, suppose that the claim is true for $n = k \geq 2$ and consider it for $n = k + 1$. Again by Step 1, we have

$$\{\omega \in \Omega \setminus N : Y_1 \leq y_1, \ldots, Y_{k+1} \leq y_{k+1}\} \supseteq \{\omega \in \Omega \setminus N : Y_i \leq y_i, \ldots, Y_{k+1} \leq y_{k+1}\} \cap \{\omega \in \Omega \setminus N : Y_{k+1} \leq y_{k+1}\} \supseteq \{\omega \in \Omega \setminus N : Y_i \leq y_i, \ldots, Y_{k+1} \leq y_{k+1}\},$$

for some $i \in \{1, \ldots, k\}$.
To show the comonotonicity of \( Y \) on \( \mathbb{U}(\mathbb{d}^*) \), we construct a new random vector \( \mathbf{Y} \) by restricting \( Y \) to the set \( \Omega = \{ \omega \in \Omega \setminus N : Y(\omega) \in \mathbb{U}(\mathbb{d}^*) \} \). Then by (4), \( P(\tilde{\Omega}) = 1 - F_Y(d^*) + P(Y = d^*) \). Note that \( P(\tilde{\Omega}) > 0 \) by (5).

Now we consider the new probability space \( (\hat{\Omega}, \hat{\mathcal{F}}, \hat{P}) \) where \( \hat{\mathcal{F}} = \mathcal{F} \cap \tilde{\Omega} \) and \( P(A) = P(A)/P(\tilde{\Omega}) \) for \( A \in \mathcal{F} \). To obtain the comonotonicity of \( \mathbf{Y} = \mathbf{Y} \bigg| \hat{\Omega} \), it suffices to check

\[
F_Y(y_1, \ldots, y_n) = \min_{1 \leq i \leq n} F_{\tilde{Y}_i}(y_i), \quad \forall (y_1, \ldots, y_n) \in \mathbb{R}^n. \tag{12}
\]

Since the interior of \( R(\mathbb{d}^*) \) has no intersection with \( \{ Y(\omega) : \omega \in \Omega \setminus N \} \) by (4), if \( y_i \geq d^* \), then

\[
F_{\tilde{Y}_i}(y_i) = P((Y_i \leq y_i) \cap \tilde{\Omega})/P(\tilde{\Omega}) = (1 - (1 - P(\tilde{\Omega}))/P(\tilde{\Omega}).
\]

Therefore, the marginal distribution function of \( \tilde{Y}_i \) is given by

\[
F_{\tilde{Y}_i}(y_i) = \begin{cases} 
0, & y_i < d^*, \\
(0, (1 - (1 - P(\tilde{\Omega}))/P(\tilde{\Omega}), & y_i \geq d^*. 
\end{cases}
\]

Similarly, for any \( y \in \mathbb{R}^n \) we have

\[
F_Y(y) = \begin{cases} 
0, & y \notin \mathbb{U}(\mathbb{d}^*), \\
(0, (1 - (1 - P(\tilde{\Omega}))/P(\tilde{\Omega}), & y \in \mathbb{U}(\mathbb{d}^*).
\end{cases}
\]

Finally, by (11) we obtain that (12) holds true and, hence, \( \mathbf{Y} \) is comonotonic on \( \mathbb{U}(\mathbb{d}^*) \).

\section*{4. Application}

In this section, we consider an application to the computation of Haezendonck risk measures. The concept of Haezendonck risk measure was first raised by Haezendonck and Goovaerts (1982) as a premium calculation principle induced by an Orlicz norm, it is multiplicatively analogous to the zero utility principle. In Goovaerts et al. (2004), the Haezendonck risk measure was derived as a translation-invariant minimal Orlicz risk measure which preserves convex order. In Bellini and Rosazza Gianin (2008a), an alternative formulation of the Haezendonck risk measure was introduced to guarantee coherence. For application to portfolio optimization, several estimators of the Haezendonck risk measure were provided by numerical simulations in Bellini and Rosazza Gianin (2008b).

\begin{definition}[Goovaerts et al., 2004] For a risk variable \( X \), \( 0 < q < 1 \) and a nonnegative, strictly increasing, and continuous function \( \psi(\cdot) \) satisfying \( \psi(0) = 0 \), \( \psi(1) = 1 \), \( \psi(+\infty) = +\infty \), the Haezendonck risk measure is defined as

\[
\pi_q[X] = \inf_{-\infty < c < \max(X)} \pi_q[X, x],
\]

where \( \pi_q[X, x] \) is the unique solution of the equation

\[
\mathbb{E} \left[ \phi \left( \frac{(X - x)^+}{\pi - x} \right) \right] = 1 - q.
\]

\end{definition}

As its definition indicates, the computation of the Haezendonck risk measure \( \pi_q[X] \) is complicated in general except for some ideal choices of \( \phi(\cdot) \) and the distribution of \( X \).

\begin{example} Suppose that \( \phi(t) = t \). For any given marginal distributions \( F_1, \ldots, F_n \), if \( Y = (Y_1, \ldots, Y_n) \in \mathbb{R}(F_1, \ldots, F_n) \) is maximal in tail convex order, then there exists some \( q^* \in (0, 1) \) such that

\[
\pi_q[Y_1 + \cdots + Y_n] = \sum_{i=1}^n \text{TVaR}_q[Y_i], \quad \forall q \in (q^*, 1). \tag{15}
\]

\end{example}

\begin{proof} By definition there exists some \( d^* \) such that \( \mathbb{E}[S^*-x^*] = \mathbb{E}[S - x^*] \) for all \( x \geq d^* \). Thus, \( \pi_q[S, x] = \pi_q[S^*, x] \) for all \( x \geq d^* \) since

\[
\pi_q[S, x] = \frac{\mathbb{E}[S - x^*]}{1 - q} + x.
\]

Moreover, if \( x \geq d^* \), then \( F_q(x) = F_{S^*}(x) \) by (ii) and (iii) of \textit{Theorem 2.3}. Indeed, the support of \( Y \) coincides with that of \( Y^* \) on \( \mathbb{U}(\mathbb{d}^*) \setminus \mathbf{d}^* \) and both of \( Y \) and \( Y^* \) have no support on \( \text{int}(R(\mathbb{d}^*)) \).

The value of the Haezendonck risk measure \( \pi_q[S] \) in (13) is attained at \( x^* \) at which the derivative

\[
\frac{d}{dx} \pi_q[S, x] = \frac{F_q(x) - q}{1 - q}
\]

changes sign. A subtle analysis shows that \( x^* = F_{S^*}^{-1}(q) \). Therefore, if \( q > q^* \equiv F_q(d^*) \), then

\[
x^* = F_{S^*}^{-1}(q) = F_{S^*}^{-1}(q) \geq d^*
\]

and

\[
\pi_q[S] = \pi_q[S, x^*] = \pi_q[S^*, x^*] = \frac{\mathbb{E}[S^*-F_{S^*}^{-1}(q)]}{1 - q} + F_{S^*}^{-1}(q).
\]

Hence, the desired relation (15) holds.

In fact, the proof of \textit{Example 4.2} shows that if \( \phi(t) = t \) then \( \pi_q[S] = \text{TVaR}_q[S] \), \( \forall q \in (0, 1) \).

See e.g. Goovaerts et al. (2004).

For a more general \( \phi(\cdot) \) of the form \( \phi(t) = t^r \) with \( r > 1 \), it is unlikely that we can establish a transparent expression for the Haezendonck risk measure. Nevertheless, for some special cases we can do so under the help of the following lemma.

\begin{lemma} If \( \phi(t) = t^r \) with \( r > 1 \) and \( q < 1 \), then \( \frac{d}{dx} \pi_q[X, x] \) is non-decreasing in \( x \) and

\[
\lim_{x \to -\infty} \frac{d}{dx} \pi_q[X, x] = 1. \tag{16}
\]

\end{lemma}

\begin{proof} For \( \phi(t) = t \), as shown in \textit{Example 4.2},

\[
\frac{d}{dx} \pi_q[X, x] = \frac{F_q(x) - q}{1 - q},
\]

which is non-decreasing and (16) holds true.

For \( \phi(t) = t^r \) with \( r > 1 \), we have

\[
\frac{d}{dx} \pi_q[X, x] = 1 - \left( \frac{1}{1 - q} \right) \frac{1}{2} \frac{d}{dx} \mathbb{E}[S^*-x^*] \frac{d}{dx} \mathbb{E}[S - x^*].
\]

Since \( r > 1 \), an application of Hölder’s inequality shows that
\[
\frac{d^2}{dx^2} \pi_q[x, x] = -\left( \frac{1}{1 - q} \right)^{\frac{1}{r - 1}} (r - 1) \left( \mathbb{E}(X - x)^{r - 2} \right) 
\times \left( \mathbb{E}(X - x)^{r - 1} \right)^2 - \mathbb{E}(X - x)^{r - 2} \right) 
\geq 0.
\]

Indeed,
\[
\mathbb{E}(X - x)^{r - 1} = \mathbb{E} \left[ (X - x)^{r - 1} \right] 
\leq \mathbb{E}(X - x)^{r - 1} \left( 1 - \mathbb{E}(X - x) \right)^{r - 2}.
\]

Thus, \( \frac{d}{dx} \pi_q[x, x] \) is continuous and non-decreasing.

Finally, by Hölder's inequality again we have
\[
\mathbb{E}(X - x)^{r - 1} = \mathbb{E} \left[ (X - x)^{r - 1} \right] 
\leq \mathbb{E}(X - x)^{r - 1} \left( 1 - \mathbb{E}(X - x) \right)^{r - 1},
\]

which, together with (17), implies (16) and the proof ends. \( \square \)

An upper comonotonic random vector \( Y = (Y_1, \ldots, Y_n) \) has a stochastic representation
\[
(Y_1, \ldots, Y_n) \overset{d}{=} F_1^{-1}(U_1), \ldots, F_n^{-1}(U_n),
\]
where \( U_i = U_1(0, \beta) + \beta V_i(0, 1) \) for some \( \beta \in [0, 1] \) and \( V_i \)'s are uniform on \((0, 1)\) and \( U \) is independent of \((V_1, \ldots, V_n)\) (see e.g., Cheung, 2009).

**Example 4.4.** Suppose that \( \phi(t) = t^r \) with \( r \geq 1 \). If the marginal distributions are exponential with rates \( \lambda_1, \ldots, \lambda_n \), respectively, then
\[
\pi_q[S] = \frac{1}{\lambda} \ln \left( \frac{1 - \beta}{r} \right) \frac{1}{\Gamma(r + 1)} + \frac{r}{\lambda} \tag{18}
\]
for \( q > 1 - \frac{1}{r} \) and \( \lambda = \left( \sum_{i=1}^{n} \frac{1}{\lambda_i} \right)^{-1} \).

**Proof.** We have
\[
\mathbb{E}(S - x)^{r - 1} = \mathbb{E} \left[ (S - x)^{r - 1} \left( 1 - \beta \right) \right] 
= \mathbb{E} \left[ \left( \frac{1}{\lambda} \ln \left( \frac{1 - \beta}{r} \right) \frac{1}{\Gamma(r + 1)} \right) \right] 
+ \beta \mathbb{E} \left[ \left( \sum_{i=1}^{n} \frac{1}{\lambda_i} \right) - x \right]^{r - 1}.
\]

If \( \lambda x \geq - \ln(1 - \beta) \), then it reduces to
\[
\mathbb{E}(S - x)^{r - 1} = \int_{\beta}^{1} \left( \frac{1}{\lambda} \ln \left( \frac{1 - \beta}{r} \right) \frac{1}{\Gamma(r + 1)} \right) \frac{\lambda}{\lambda - x} \frac{r}{\Gamma(r + 1)}
\]
so that
\[
\frac{d}{dx} \pi_q[S, x] = 1 - \frac{1}{r} \left( \frac{1}{\lambda} \ln \left( \frac{1 - \beta}{r} \right) \frac{1}{\Gamma(r + 1)} \right) e^{-\lambda x}.
\]

By Lemma 4.3, if \( \frac{d}{dx} \pi_q[S, x] \bigg|_{x = \ln \left( \frac{1 - \beta}{r} \right)} < 0 \), or, equivalently,
\[
q > 1 - \frac{1}{r} \frac{\Gamma(r + 1)}{1 - \beta},
\]
then the equation \( \frac{d}{dx} \pi_q[S, x] = 0 \) has a solution
\[
x^* = \frac{1}{\lambda} \ln \left( \frac{1 - \beta}{r} \right) \frac{1}{\Gamma(r + 1)} - \frac{1}{\lambda}.
\]

Thus, we obtain the desired Haefendornck risk measure (18) as \( \pi_q[S] = \pi_q[S, x^*] \). Note that if \( \beta = 0 \), then this example reduces to the case of complete comonotonicity. \( \square \)

### 5. Conclusion

The contribution of this paper is twofold. First, we showed a characterization of upper comonotonicity via tail convex order. Second, as an application, we considered the computation of the Haefendornck risk measure of the sum of upper comonotonic random variables and we established a transparent expression for the case with exponential marginal distributions and a large confidence level.

Note that upper comonotonicity corresponds to a special case of asymptotic dependence in the upper tail with coefficient \( 1 \). It is desirable to extend some applications of upper comonotonicity to the case of asymptotic dependence. We shall pursue such extensions in our future research.

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### Appendix

It is well known that a random vector \( Y = (Y_1, \ldots, Y_n) \) is comonotonic if and only if it is pairwise comonotonic (see e.g., Dhaene et al., 2002a). This is, however, no longer true for an upper comonotonic random vector.

**Example A.1.** For \( n = 3 \) and \( \Omega = \{\omega_1, \ldots, \omega_8\} \), we define a random vector \( Y = (Y_1, Y_2, Y_3) \) as follows:
\[
\begin{array}{ccc|c}
\omega & Y_1 & Y_2 & Y_3 & P \\
\hline
\omega_1 & -2 & -1 & 0 & 1/8 \\
\omega_2 & -1 & -1 & 0 & 1/8 \\
\omega_3 & -1 & -2 & 0 & 1/8 \\
\omega_4 & -1 & 1 & -1 & 1/8 \\
\omega_5 & 0 & -1 & 1 & 1/8 \\
\omega_6 & 0 & 0 & 2 & 1/8 \\
\omega_7 & 1 & 1 & 2 & 1/8 \\
\omega_8 & 2 & 1 & 2 & 1/8 \\
\end{array}
\]

It is easy to check that each pair \((Y_i, Y_j)\) is upper comonotonic. Indeed, the thresholds of \((Y_1, Y_2), (Y_2, Y_3), (Y_3, Y_1)\) are \((0, 0), (-1, 1), \) and \((0, -1), \) respectively. Note, however, that the comonotonicity is broken down due to \(\omega_{20} \) and \(\omega_{21} \). Since \(Y(\omega_3)\) and \(Y(\omega_{20})\) always have at least one component with the same values for \( i = 1, \ldots, 7 \) like a chain, there is no point \( d \in \mathbb{R}^3 \) satisfying the three conditions in Definition 2.1. Therefore, \( Y \) is not upper comonotonic.

The following example illustrates the necessity of the condition \( P(Y \in \text{bd}(U(d^*))) = 0 \) in Theorem 2.3 for \( Y \) to be upper comonotonic.

**Example A.2.** For \( n = 2 \) and \( \Omega = \{\omega_1, \ldots, \omega_{10}\} \), we define a random vector \( Y = (Y_1, Y_2) \) with the corresponding comonotonic counterpart \((Y^*_1, Y^*_2)\) as follows:
\[
\begin{array}{ccc|c}
\omega & Y_1 & Y_2 & Y_1 + Y_2 & P \\
\hline
\omega_1 & -3 & -2 & 1 & 1/10 \\
\omega_2 & -2 & -2 & 0 & 1/10 \\
\omega_3 & -3 & 0 & -3 & 1/10 \\
\omega_4 & -2 & -1 & -3 & 1/10 \\
\omega_5 & -1 & -2 & 1/10 \\
\omega_6 & 0 & -1 & 1/10 \\
\omega_7 & 0 & 2 & 1/10 \\
\omega_8 & 1 & 2 & 3 & 1/10 \\
\omega_9 & 3 & 2 & 5 & 1/10 \\
\end{array}
\]
It is easy to check that the stop-loss premiums of $Y_1 + Y_2$ and $Y_1^c + Y_2^c$ are equal for all $d \geq 0$, i.e.,

$$E[(Y_1^c + Y_2^c - d)_+] = E[(Y_1 + Y_2 - d)_+] \quad \forall d \geq 0.$$

Although the support of $(Y_1, Y_2)$ is comonotonic on $U(0, 0)$, $(Y_1, Y_2)$ is not upper comonotonic. The reason exists in that $P((Y_1, Y_2) \in \text{bd}(U(0, 0))) = \frac{1}{10} > 0$.

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\begin{table}[h]
\centering
\begin{tabular}{cccccc}
\hline
$\omega$ & $Y_1^c$ & $Y_2^c$ & $Y_1^c + Y_2^c$ & P \\
\hline
$\omega_1$ & $-3$ & $-2$ & $-5$ & 1/10 \\
$\omega_2$ & $-3$ & $-2$ & $-5$ & 1/10 \\
$\omega_3$ & $-2$ & $-1$ & $-3$ & 1/10 \\
$\omega_4$ & $-2$ & $-1$ & $-3$ & 1/10 \\
$\omega_5$ & $-1$ & $-1$ & $-2$ & 1/10 \\
$\omega_6$ & $-1$ & $0$ & $-1$ & 1/10 \\
$\omega_7$ & $0$ & $0$ & $0$ & 1/10 \\
$\omega_8$ & $0$ & $2$ & $2$ & 1/10 \\
$\omega_9$ & $1$ & $2$ & $3$ & 1/10 \\
$\omega_{10}$ & $3$ & $2$ & $5$ & 1/10 \\
\hline
\end{tabular}
\end{table}

References