Interplay of Insurance and Financial Risks with Bivariate Regular Variation

Qihe Tang
*University of Iowa, United States*

Zhongyi Yuan
*The Pennsylvania State University, United States*

Abstract

It is known that for an insurer who invests in the financial market, the financial investments may affect its solvency as severely as do insurance claims. This conclusion is usually reached under an assumption of independence or asymptotic independence between insurance risk and financial risk. Such an assumption seems reasonable if the insurer focuses on the traditional insurance business that does not interact much with the capital market. However, we shall argue that at least for insurers who participate in financial guarantee insurance and as a result cause systemic risk, asymptotic dependence between insurance and financial risks needs to be considered. Under a bivariate regular variation structure, we investigate the interplay of insurance and financial risks, and show that the asymptotic dependence introduces extra risk for the insurer’s solvency.

20.1 Introduction

In the aftermath of the 2008 financial crisis, the then Federal Reserve Chairman Ben Bernanke once commented on the bailout of AIG: “AIG . . . was a hedge fund, basically, that was attached to a large and stable insurance company.” By stressing its nature as a hedge fund, Mr. Bernanke was accusing AIG of making large amounts of aggressive investments and “irresponsible bets.” In fact, many other insurance companies (the monolines, in particular) also engaged in such investments, which
greatly contributed to the financial crisis. Such aggressive investments were able to get around regulation due to the huge gap in the regulation system, which today’s risk management is dedicated to fill. In the meantime, risk management also calls for better understanding of the risks the insurance industry faces. In this chapter, we shall discuss additional risk introduced to insurance companies by such an investment behavior, and shall pay particular attention to extreme risks that may cause bankruptcy for the insurance companies.

Typical risks that can affect an insurer’s survival include large claims from catastrophic events and large losses from financial investments. For example, more than a dozen insurers became insolvent because of claims originating from the 2005 Hurricane Katrina (e.g., Muermann, 2008), and, as discussed above, a large number of insurers experienced financial distress or even collapsed due to their high exposure to mortgage-related securities (e.g., Harrington, 2009). Research also confirms that risky investments may impair the insurer’s solvency just as severely as do large claims (e.g., Frolova et al., 2002; Kalashnikov and Norberg, 2002; Li and Tang, 2013; Norberg, 1999; Pergamenshchikov and Zeitouny, 2006; Tang and Tsitsiashvili, 2003). In the literature, the risk resulting from insurance claims and the risk from financial investments are generally referred to as insurance risk and financial risk, respectively, although general financial risk can be further categorized to market risk, credit risk, operational risk, and so on.

We shall describe the insurance business by a discrete-time risk model in which the two risks are quantified by concrete random variables. The probability of ruin in both finite-time and infinite-time horizons is then studied. This study has become particularly relevant for insurance because of modern regulatory frameworks (such as EU Solvency II) that require insurers to hold solvency capital so that the ruin probability is under control. Studies of the asymptotic behavior of the ruin probability of this model have been done by many authors (Chen, 2011; Li and Tang, 2013; Nyren, 1999, 2001; Tang and Tsitsiashvili, 2003, 2004; Yang et al., 2012; Yang and Wang, 2013; Zhou et al., 2012).

A continuous-time analogy is the so-called bivariate Lévy-driven risk model, proposed by Paulsen (1993), in which the cash flow of premiums less claims is described as a Lévy process while the log price of the investment portfolio as another Lévy process. See a survey paper by Paulsen et al. (2008) for the study of the ruin probability of this risk model or one of its ramifications.

Thus far, it is commonly assumed that the insurance and financial risks are independent or asymptotically independent. Such an assumption is usually justified in the literature by virtue of the insufficient link between the traditional/core insurance business and the capital market. However, due to the recent changes in the insurance business, we have at least two reasons to believe that the two risks should be dependent or asymptotically dependent for certain insurers. First, to hedge against catastrophic risk, more and more insurers now securitize their insurance risk and export it to the capital market using insurance-linked securities, such as catastrophe bonds. Therefore, an insurer who invests in the capital market is likely to be exposed to the insurance risk exported by another insurer, yielding interconnectedness between his/her insurance and financial risks. Second, it has been argued in the recent
literature of insurance and finance that, unlike the core insurance activities, noncore insurance activities such as financial guarantee insurance cause systemic risk, yielding another source of interconnectedness in extreme situations. For example, situations like the recent Great Recession would make insurers who have participated in both default protection insurance and the distressed financial market, such as AIG and the monolines, suffer losses from both sides. The more and more involvement of insurers in the noncore insurance business has resulted in significant systemic risk. Therefore, at least for insurers engaged in noncore activities, it is reasonable to assume that insurance claims and investment losses exhibit asymptotic dependence. For related discussions, we refer the reader to Acharya (2009), Bell and Keller (2009), Baluch et al. (2011), Billio et al. (2012), and Cummins and Weiss (2014).

To quantify the extreme insurance and financial risks impairing the insurer’s solvency, it is necessary to model both their enormous sizes and extreme dependence. We shall show that this can be done in a unified framework of bivariate regular variation developed in multivariate extreme value theory. We shall characterize the insurance and financial risks through a bivariate regular variation structure, and derive a unified asymptotic approximation for the probability of ruin in both finite-time and infinite-time horizons. Our result reinforces the longstanding viewpoint that the asymptotic dependence between insurance claims and financial investment losses introduces substantial risk to the insurer’s solvency.

The rest of this chapter is organized as follows: In Section 20.2 a discrete-time risk model is introduced as a platform for insurance and financial risks and some numerical studies are conducted to explore the asymptotic behavior of the ruin probability; the main result is presented in Section 20.3 after a brief review of bivariate regular variation; in Section 20.4 two more numerical examples are provided to examine the performance of the asymptotic formula; and finally, Section 20.5 completes the chapter with a proof of the main result.

### 20.2 Ruin under the Interplay of Insurance and Financial Risks

Throughout this chapter, the following notational conventions are in force. An operation \(*\) between a scalar number \(c\) and a vector \((x, y)\) yields a vector \((c * x, c * y)\). For a number \(c \in \mathbb{R}\) and a set \(A \subset \mathbb{R}^2\), write the positive part of \(c\) as \(c^+ = c \vee 0\) and the set \(\{cx : x \in A\}\) as \(cA\). Moreover, for two positive functions \(f(\cdot)\) and \(g(\cdot)\), we write \(f(\cdot) \sim g(\cdot)\) if \(\lim f(\cdot)/g(\cdot) = 1\), write \(f(\cdot) \preceq g(\cdot)\) if \(\limsup f(\cdot)/g(\cdot) \leq 1\), and write \(f(\cdot) \succeq g(\cdot)\) if \(\liminf f(\cdot)/g(\cdot) \geq 1\). For a nondecreasing function \(h(\cdot)\), define its generalized inverse as \(h^-(y) = \inf \{x \in (-\infty, \infty) : h(x) \geq y\}\), where we follow the usual convention \(\inf \emptyset = \infty\).
20.2.1 A Discrete-Time Risk Model

Denote by $W_0 = x > 0$ the deterministic initial wealth of the insurer, and by $W_k$ the insurer’s wealth at time $k \in \mathbb{N}$. The insurer’s realized net profit from the insurance business is roughly equal to premiums collected, denoted by $P_k$, minus costs resulting from insurance claims and other expenses, denoted by $Z_k$. Suppose that, at the beginning of each period $k$, the insurer invests its wealth $W_{k-1}$ into risk-free and risky assets that result in an overall return rate $R_k \in [-1, \infty)$, so that the insurer gets back $(1 + R_k)W_{k-1}$ at the end of the period. The wealth process $\{W_k, k \in \mathbb{N}\}$ evolves according to

$$W_k = (1 + R_k)W_{k-1} + (P_k - Z_k), \quad k \in \mathbb{N}.$$ 

Iterating this yields

$$W_k = x \prod_{j=1}^{k} (1 + R_j) + \sum_{i=1}^{k} (P_i - Z_i) \prod_{j=i+1}^{k} (1 + R_j),$$

where multiplication over an empty index set produces value 1 by convention.

For $n \in \mathbb{N} \cup \{\infty\}$, we consider the probability of ruin in either finite-time or infinite-time horizon $n$, defined by

$$\psi(x; n) = \Pr\left( \min_{0 \leq k < n+1} W_k < 0 \left| W_0 = x \right. \right).$$

Following Tang and Tsitsiashvili (2003), introduce

$$X_i = Z_i - P_i \quad \text{and} \quad Y_i = \frac{1}{1 + R_i}, \quad i \in \mathbb{N},$$

to quantify the insurance risk and financial risk, respectively. Then we can rewrite the ruin probability as

$$\psi(x; n) = \Pr\left( \min_{0 \leq k < n+1} \prod_{j=1}^{k} (1 + R_j) \left( x + \sum_{i=1}^{k} (P_i - Z_i) \prod_{j=i+1}^{k} \frac{1}{1 + R_j} \right) < 0 \right)$$

$$= \Pr\left( \max_{1 \leq k < n+1} \sum_{i=1}^{k} X_i \prod_{j=1}^{i} Y_j > x \right). \quad (20.1)$$

This way $\psi(x; n)$ reduces to the tail probability of the running maximum of a sequence of quantities with a sum-product structure.

20.2.2 Explorative Numerical Studies

To obtain a rough idea on how the asymptotic dependence between the insurance risk and financial risk would affect the insurer’s solvency, we conduct two simple
Interplay of Insurance and Financial Risks with Bivariate Regular Variation

numerical studies to compare the ruin probability under the asymptotic dependence assumption with that under the independence assumption. For all our numerical studies in this chapter, we further assume that the nonnegative random vectors \((Z_i, Y_i), i \in \mathbb{N}\), form a sequence of independent and identically distributed (i.i.d.) random pairs with generic pair \((Z, Y)\). For simplicity, the premium amount collected during each period is chosen to be a constant yielding a safety loading of about 10\%, and the numerical studies are all conducted for the probability of ruin in time horizon \(n = 3\) with initial capital \(x\) ranging from 50 to 4000.

**Numerical study 1** Let \(Z\) and \(Y\) both follow a Pareto distribution \(F\) with shape parameter \(\alpha\) and scale parameter \(\theta\); that is,

\[
F(x) = 1 - \left(\frac{\theta}{x + \theta}\right)^\alpha, \quad x > 0. \tag{20.2}
\]

The parameters are set to be \(\alpha = 2\) and \(\theta = 1\), respectively. For the asymptotically dependent case, the dependence structure of \(Z\) and \(Y\) is described by a Gumbel copula.

\[
C(u, v) = \exp\left\{ -(( - \ln u)^r + (- \ln v)^r)^{1/r}\right\}, \quad (u, v) \in (0, 1)^2. \tag{20.3}
\]

It is known that a Gumbel copula with \(r > 1\) produces asymptotic dependence with coefficient \(2 - 2^{1/r}\), so a larger value of \(r\) means a stronger dependence in the tail area (e.g., McNeil et al., 2005, p. 209). The parameter \(r\) is set to be 2 or 10 in order to further demonstrate the effect of dependence strength. Samples of \((Z, Y)\) following a Gumbel copula are generated in the \(R\) environment using the `copula` package (Yan, 2007).

Assume that \(P_k \equiv (1 + 10\%)E[Z], k = 1, \ldots, n\). For both the asymptotically dependent case and the independent case, we simulate \(N = 10^6\) samples of 
\[
((X_1, Y_1), \ldots, (X_n, Y_n))
\]
and use the empirical estimator to estimate the ruin probability in relation (20.1); that is, we estimate \(\psi(x; n)\) by

\[
\frac{1}{N} \sum_{m=1}^{N} 1_{\left\{\max_{1 \leq k < n+1} \sum_{i=1}^{k} X_i^{(m)} \Pi_{j=1}^{1} Y_j^{(m)} > x\right\}}, \tag{20.4}
\]

where 
\[
\left(\left( X_1^{(m)}, Y_1^{(m)} \right), \ldots, \left( X_n^{(m)}, Y_n^{(m)} \right)\right)\] for each \(m = 1, \ldots, N\) is an independent copy of \((X_1, Y_1), \ldots, (X_n, Y_n))\), and \(1_{\cdot}\) is the indicator function. Figure 20.1 shows the ratios of the estimated ruin probability under the independent case to that under the asymptotically dependent case. We observe that the ratios are smaller than 1, and quickly decay to 0 as \(x\) increases, meaning that the ruin probability under the asymptotically dependent case is significantly higher.

**Numerical study 2** Next we investigate the case in which the nonnegative generic pair \((Z, Y)\) follows a mixture structure
FIGURE 20.1
Independent case versus asymptotically dependent (Gumbel) case.

\[
\begin{align*}
Z &= \left[ m_Z + \sqrt{W} \left( \sqrt{\rho \eta_0} + \sqrt{1 - \rho \eta_Z} \right) \right]^+, \\
Y &= \left[ m_Y + \sqrt{W} \left( \sqrt{\rho \eta_0} + \sqrt{1 - \rho \eta_Y} \right) \right]^+,
\end{align*}
\]

(20.5)

where \( m_Z \in \mathbb{R}, m_Y \in \mathbb{R} \) and \( 0 < \rho < 1 \) are nonrandom, \( \eta_0, \eta_Z, \) and \( \eta_Y \) are i.i.d. standard normal random variables, and \( W \), independent of \( \{ \eta_0, \eta_Z, \eta_Y \} \), follows an inverse gamma distribution with both shape and scale parameters equal to \( \alpha/2 \); that is, \( W \) has probability density function

\[
f_W(w) = \left( \frac{\alpha}{2} \right)^{\alpha/2} \frac{1}{\Gamma(\alpha/2)} w^{-(\alpha/2+1)} \exp \left\{ -\frac{\alpha}{2w} \right\}, \quad w > 0.
\]

It is known that relation (20.5) yields an asymptotically dependent structure between \( Z \) and \( Y \) (e.g., McNeil et al. 2005, Section 5.3).

For comparison, we also consider the independent counterparty of (20.5) given by

\[
\begin{align*}
Z &= \left[ m_Z + \sqrt{W_Z} \left( \sqrt{\rho \eta_0} + \sqrt{1 - \rho \eta_Z} \right) \right]^+, \\
Y &= \left[ m_Y + \sqrt{W_Y} \left( \sqrt{\rho \eta_0} + \sqrt{1 - \rho \eta_Y} \right) \right]^+,
\end{align*}
\]

(20.6)

where \( W_Z \) and \( W_Y \) are independent copies of \( W \), and the other modeling components are specified the same as in relation (20.5).

We set the parameters to be \( \alpha = 2, m_Z = 10, m_Y = 0, \rho = 0.5 \) or \( 0.9 \). The premium collected within each period is set to be \( P_k \equiv 11, k = 1, \ldots, n \).
Again, with $N = 10^6$ samples, we use the empirical estimator \eqref{eq:empirical} to estimate the ruin probability $\psi(x; n)$ for both the independent case \eqref{eq:independent} and the asymptotically dependent case \eqref{eq:asymptotic}. The ratios of the ruin probability under the independent case to that under the asymptotically dependent case are demonstrated in Figure \ref{fig:20.2}, which also shows rapid decays to 0.

The two explorative numerical studies above suggest that the decay rate of $\psi(x; n)$ with respect to $x$ in the independent case is much faster than that in the asymptotically dependent case. More about this can be revealed via a rough quantitative analysis, for which we need the concept of regular variation. Recall that a positive measurable function $h(\cdot)$ on $\mathbb{R}^+ = [0, \infty)$ is said to be regularly varying at $\infty$ with regularity index $\alpha \in \mathbb{R}$ if

$$\lim_{x \to \infty} \frac{h(xy)}{h(x)} = y^\alpha, \quad y > 0.$$  

In this chapter, we shall often assume for a distribution function $F$ that its tail $\overline{F} = 1 - F$ is regularly varying. The reader is referred to Bingham et al. (1987) or Resnick (1987) for a comprehensive treatment of regular variation.

Back to the ruin probability $\psi(x; n)$. For the independent case, by Corollary 2.1 of Chen and Xie (2005) we have

$$\psi(x; n) \sim \sum_{i=1}^{n} \Pr \left( X_i \prod_{j=1}^{i} Y_j > x \right).$$
Then applying the Corollary on Page 245 of Embrechts and Goldie [1980] we know that $\psi(\cdot; n)$ is regularly varying with index $-\alpha$. However, for the asymptotically dependent case we have

$$\psi(x; n) \geq \psi(x; 1) = \Pr(X_1 Y_1 > x).$$

For the special case where $Z = Y$, it is easy to show that $\Pr(X_1 Y_1 > x) \sim F(\sqrt{x})$, and hence, $\psi(x; n)$ is bounded below by a positive function which is regularly varying with index $-\alpha/2$. This confirms what we have observed above and also motivates us to conduct the numerical study below.

**Numerical study 3** Based on the setup of the two numerical studies above, we conduct two more simulations to compare $\psi(x; n)$ with $F(\sqrt{x})$.

We see that the dots in Figure 20.3 and Figure 20.4 appear to be roughly on a straight line around the lower-left corner, which corresponds to larger values of $x$. This means that for $x$ large, the ruin probability $\psi(x; n)$ decays roughly at rate $F(\sqrt{x})$ under both the Gumbel case and the mixture case. Also, we observe that a stronger dependence between $X$ and $Y$ in the tail area leads to a larger ruin probability, which is no surprise. These observations will be theoretically verified in our main result below.
20.3 Main Result

20.3.1 Bivariate Regular Variation

First we give a brief review of bivariate regular variation, which is the two-dimensional version of the concept of Multivariate regular variation (MRV). Since its introduction by De Haan and Resnick (1981), the MRV framework has found its use in insurance, finance, and risk management by providing a tool for modeling extreme risks with both heavy tails and asymptotic (in)dependence. Recent works include Embrechts et al. (2009), Böcker and Klüppelberg (2010), Mainik and Rüschendorf (2010), Joe and Li (2011), Asimit et al. (2011), Part IV of Rüschendorf (2013), Tang and Yuan (2013), among many others. In particular, Fougères and Mercadier (2012) studied the ruin probability in a broader scope using an MRV structure with asymptotic dependence.

A random vector \((\xi_1, \xi_2)\) taking values in \([0, \infty]^2 \setminus \{0\}\) is said to follow a distribution with a bivariate regularly varying tail if there exist a positive normalizing function \(b(\cdot)\) monotonically increasing to \(\infty\) and a limit measure \(\nu\) not identically 0, such that, as \(x \to \infty\),

\[
x \Pr \left( \frac{(\xi_1, \xi_2)}{b(x)} \in \cdot \right) \xrightarrow{v} \nu(\cdot) \quad \text{on} \ [0, \infty]^2 \setminus \{0\}, \tag{20.7}
\]

where \(\xrightarrow{v}\) denotes vague convergence. Discussions on vague convergence can be
The normalizing function \( b(\cdot) \) is not unique, but different choices may result in limit measures that differ by a constant factor. See Section 5.4.2 or Section 6.1.4 of Resnick (2007) for more discussions.

The definition of bivariate regular variation by relation (20.7) implies that the limit measure \( \nu \) is homogeneous; that is, there exists some \( 0 < \alpha < \infty \), representing the bivariate regular variation index, such that \( \nu(tB) = t^{-\alpha}\nu(B) \) holds for every Borel set \( B \subset [0, \infty)^2 \setminus \{0\} \). See Page 178 of Resnick (2007) for the proof of this result. Consequently, the function \( b(\cdot) \) is regularly varying with index \( \frac{1}{\alpha} \). Thus, for some distribution function \( F \) with \( F(x) \sim 1 / b^{-1}(x) \), it is easy to see that the bivariate regular variation structure in relation (20.7) can be alternatively expressed as follows:

\[
\frac{1}{F(x)} \Pr \left( \frac{(\xi_1, \xi_2)}{x} \in \cdot \right) \overset{\nu}{\to} \nu(\cdot) \quad \text{on} \quad [0, \infty)^2 \setminus \{0\}. \tag{20.8}
\]

Hereafter, we shall follow relation (20.8) when specifying a bivariate regular variation structure.

### 20.3.2 Main Result

We assume that \( (X_i, Y_i), i \in \mathbb{N} \), form a sequence of i.i.d. random pairs with generic pair \( (X, Y) \), and that \( (X^+, Y) \) follows a bivariate regular variation structure given below:

**Assumption 20.3.1.** The vague convergence

\[
\frac{1}{F(x)} \Pr \left( \frac{(X^+, Y)}{x} \in \cdot \right) \overset{\nu}{\to} \nu(\cdot) \quad \text{on} \quad [0, \infty)^2 \setminus \{0\} \tag{20.9}
\]

holds for some auxiliary distribution function \( F \) with regularly varying tail with index \( -\alpha \), and some limit measure \( \nu \) with \( \nu([1, \infty] \times (0, \infty]) > 0 \).

Assumption 20.3.1 has some immediate implications. First, the assumption \( \nu([1, \infty] \times (0, \infty]) > 0 \) means that

\[
\lim_{x \to \infty} \frac{1}{F(x)} \Pr (X > x, Y > x) = \nu([1, \infty] \times [0, \infty]) > 0,
\]

which indicates that \( X \) and \( Y \) exhibit large joint movements, and hence, are asymptotically dependent. Second, relation (20.9) also implies that

\[
\lim_{x \to \infty} \frac{\Pr (X > x)}{\Pr (Y > x)} = \frac{\nu([1, \infty] \times [0, \infty])}{\nu([0, \infty] \times [1, \infty])},
\]

indicating that the two risk variables \( X \) and \( Y \) have comparable tails. Third, introducing a set \( A = \{(s, t) \in [0, \infty]^2 : st > 1\} \), it is easy to verify that \( \nu(\partial A) = 0 \) and that \( \nu(A) > 0 \). Thus, Assumption 20.3.1 leads to

\[
\Pr (XY > x) = \Pr \left( \frac{(X, Y)}{\sqrt{x}} \in A \right) \sim \nu(A)F\left(\sqrt{x}\right); \tag{20.10}
\]
see also Proposition 7.6 of [Resnick (2007)] for a more general discussion. This shows that the product \( XY \) has a regularly varying tail with index \(-\alpha/2\).

**Theorem 20.3.1.** Under Assumption 20.3.1, 

(a) it holds for every \( n \in \mathbb{N} \) that

\[
\psi(x; n) \sim \left( \sum_{i=1}^{n} \left( \mathbb{E} \left[ \frac{Y^{\alpha/2}}{2} \right] \right)^{i-1} \right) \nu(A) F \left( \sqrt{x} \right); \tag{20.11}
\]

(b) if further \( \mathbb{E} \left[ \frac{Y^{\alpha/2}}{2} \right] < 1 \), then relation (20.11) holds uniformly for all \( n \in \mathbb{N} \) and, hence,

\[
\psi(x; \infty) \sim \frac{\nu(A)}{1 - \mathbb{E} \left[ \frac{Y^{\alpha/2}}{2} \right]} F \left( \sqrt{x} \right). \tag{20.12}
\]

It is meaningful to compare Theorem 20.3.1 with the results of [Li and Tang (2015)]. In the latter paper, the authors have shown that for independent insurance and financial risks, each having a strongly regularly varying tail, the ruin probability \( \psi(x; n) \) is equivalent to a linear combination of \( \Pr(X > x) \) and \( \Pr(Y > x) \). Thus, our Theorem 20.3.1 quantitatively captures the extra risk that the asymptotic dependence between the insurance and financial risks introduces to the insurer’s solvency.

### 20.4 Accuracy Examination

In this section, we examine the accuracy of the asymptotic formula (20.11). Due to the infeasibility of simulating the infinite-time ruin probability, we skip formula (20.12). We shall demonstrate via two numerical studies that (20.11) can serve as a good approximation for the ruin probability when the initial surplus is reasonably large. Again, for simplicity we consider the probability of ruin in time horizon \( n = 3 \) with initial capital \( x \) ranging from 50 to 4000.

#### 20.4.1 Gumbel Case

Similar to Section 20.2.2, we consider the case where \( Z \) and \( Y \) both follow a Pareto distribution given by (20.2) with \( \alpha = 2 \) and \( \theta = 1 \), and their dependence structure is described by the Gumbel copula (20.3) with \( r = 2 \) or 10. Note that in this case \((X, Y)\) follows a bivariate regular variation structure on \([0, \infty)^2 \setminus \{0\}\) satisfying relation (20.9), in which the distribution function \( F \) is given by (20.2), and the limit measure \( \nu \) is defined by

\[
\nu \left[ 0, t \right]^c = \left( t_1^{-\alpha r} + t_2^{-\alpha r} \right)^{1/r}, \quad t > 0; \tag{20.13}
\]

see e.g., Lemma 5.2 of [Tang and Yuan (2013)].

We calculate the ratio of the left-hand side of (20.11) to the right-hand side. The
left-hand side is estimated using the empirical estimator (20.4), while in order to evaluate the right-hand side, we need to estimate \( \nu(A) \) for \( A = \{(s,t) \in [0,\infty]^2 : st > 1\} \) and \( \nu \) defined by (20.13). Note that the bivariate regular variation structure (20.9) implies that, as \( N \uparrow \infty \),

\[
\frac{1}{k} \sum_{i=1}^{N} \epsilon((X^{(i)},Y^{(i)})/F^{-1}(1-k/N)(\cdot)) \xrightarrow{w} \nu(\cdot),
\]

for every sequence \( k = k(N) \uparrow \infty \) with \( k(N) \sim k(N+1) \) and \( N/k \uparrow \infty \), where \( \epsilon(\cdot) \) denotes the Dirac measure, \( (X^{(i)},Y^{(i)}), i = 1, \ldots, N, \) are i.i.d. copies of \( (X,Y) \), and \( \xrightarrow{w} \) denotes weak convergence; see Theorem 6.2 of Resnick (2007) for this assertion. Therefore, a natural estimator for \( \nu(A) \) is given by

\[
\frac{1}{k} \sum_{i=1}^{N} 1((X^{(i)},Y^{(i)}) \in F^{-1}(1-k/N)A) = \frac{1}{k} \sum_{i=1}^{N} 1(\sqrt{X^{(i)}Y^{(i)}} > F^{-1}(1-k/N));
\]

see e.g., Section 9.2 of Resnick (2007) for discussions on the estimation of the limit measure of an MRV structure. The value of \( k \) is not a straightforward choice; in our numerical studies, with \( N = 10^6 \) samples, we use the algorithm described in Sections 9.2.4 and 11.2.2 of Resnick (2007) to choose a suitable value of \( k \). We refer the reader to Nguyen and Samorodnitsky (2013) and references therein for recent discussions on this topic. Notice that although the limit measure still has to be estimated via a stochastic simulation, such a simulation is no longer a rare-event simulation, unlike the one we conducted for the ruin probability, and therefore, simulation efficiency will be less of an issue.

The graphs in Figure 20.5 show that the ratio approaches 1 as \( x \) becomes large. The fluctuation for larger values of \( x \) must be due to the variation increase of the empirical estimator for the ruin probability when large values of \( x \) lead to too small values of the ruin probability; see e.g., Tang and Yuan (2012) for a detailed discussion.

### 20.4.2 Mixture Case

In this subsection, we investigate the accuracy of the asymptotic approximation (20.11) under the mixture structure given by relation (20.5) in Section 20.2.2. Note that in this case \( (X,Y) \) follows a bivariate regular variation structure on \([0,\infty]^2 \setminus \{0\}\) satisfying relation (20.9), where \( F \) is given by the distribution function of \( \sqrt{W} \), and the limit measure is given by

\[
\nu(\cdot) = \mathbb{E} \left[ \nu_W \left( (\eta^+)^{-1} \cdot \right) \right]
\]

with \( (\eta^+)^{-1} \cdot \) defined by the set of \((y_1,y_2) \in [0,\infty]^2 \setminus \{0\}\) such that

\[
\left( \left( \sqrt{\rho \eta_0} + \sqrt{1 - \rho \eta_X} \right)^+, \left( \sqrt{\rho \eta_0} + \sqrt{1 - \rho \eta_Y} \right)^+ \right) y_2 \right) \in \cdot,
\]
FIGURE 20.5
The ratio of empirical estimate to asymptotic approximation of the ruin probability — the Gumbel case.

and the limit measure \( \nu_W \) on \([0, \infty)^2 \setminus \{(0)\} \) defined by

\[
\nu_W[0, t]^c = \left( \prod_{i=1}^n t_i \right)^{-\alpha}, \quad t > 0;
\]

see e.g., Section 5.2 of Tang and Yuan (2013) for this claim. Thus, \( \nu(A) \) in relation (20.10) can be calculated as

\[
\nu(A) = E \left[ \nu_W \left( (\eta^+)\{0\} \right) \right] = E \left[ \nu_W \left\{ y > 0 : \left( \sqrt{\rho \eta_0 + 1 - \rho \eta X} \right)^+ \left( \sqrt{\rho \eta_0 + 1 - \rho \eta Y} \right)^+ y_1 y_2 > 1 \right\} \right] = E \left[ \left( \sqrt{\rho \eta_0 + 1 - \rho \eta X} \right)^+ \left( \sqrt{\rho \eta_0 + 1 - \rho \eta Y} \right)^+ \right]^{\alpha/2}.
\]

The expectation above is estimated via simulation.

Again, we estimate the ruin probability \( \psi(x; n) \) using the empirical estimator with \( N = 10^6 \) simulation samples, and compare it with the asymptotic approximation given on the right-hand side of (20.11). The parameters are set the same as in Section 20.2.2. The graphs in Figure 20.6 also indicate that the ratio approaches 1 as \( x \) becomes large.
Therefore, we may conclude that for $x$ reasonably large our asymptotic formula (20.11) can provide a good approximation for $\psi(x; n)$. Although we do have to carefully choose the parameters to present good graphs, our purpose is mainly to ensure that the ruin probability is not too small, so that the reliability of simulation with a sample size of $10^6$ is not an issue.

20.5 Proof of Theorem 20.3.1

The following lemma will be needed for the proof of Theorem 20.3.1:

**Lemma 20.5.1.** Under Assumption 20.3.1 the products $X_i \prod_{j=1}^{i} Y_j$, $i \in \mathbb{N}$, appearing in (20.1) are pairwise asymptotically independent.

**Proof.** Since $XY$ has a regularly varying tail with index $-\alpha/2$ as shown in (20.10), by Breiman’s theorem (see Breiman 1965, Cline and Samorodnitsky 1994) we have

$$
\Pr \left( X_i \prod_{j=1}^{i} Y_j > x \right) \sim \left( \mathbb{E} \left[ Y^{\alpha/2} \right] \right)^{i-1} \Pr (X_i Y_i > x) \\
\sim \left( \mathbb{E} \left[ Y^{\alpha/2} \right] \right)^{i-1} \nu(A) \mathcal{F} (\sqrt{x}).
$$

(20.14)
For arbitrarily chosen $1 \leq i_1 < i_2 < \infty$, by Lemma 6.2 of Tang and Yuan (2012)
\[
\Pr \left( X_{i_1} \prod_{j=1}^{i_1} Y_j > x, X_{i_2} \prod_{j=1}^{i_2} Y_j > x \right)
\]
\[
= \Pr \left( X_{i_1} \prod_{j=1}^{i_1} Y_j > x, (X_{i_2}Y_{i_2}) \prod_{j=1}^{i_2-1} Y_j > x \right)
\]
\[
= o(1) \Pr (X_{i_2}Y_{i_2} > x)
\]
\[
= o(1) F(\sqrt{x}) .
\]
By relation (20.14), this joint tail probability is $o(1)$ of each individual tail probability, indicating asymptotic independence between $X_{i_1} \prod_{j=1}^{i_1} Y_j$ and $X_{i_2} \prod_{j=1}^{i_2} Y_j$.
This concludes the proof of Lemma 20.5.1.

**Proof of Theorem 20.3.1**
(a) By Lemma 20.5.1 above, Theorem 3.1 of Chen and Yuen (2009), and relation (20.14),
\[
\psi(x; n) \leq \Pr \left( \sum_{i=1}^{n} X_i^+ \prod_{j=1}^{i} Y_j > x \right)
\]
\[
\sim \sum_{i=1}^{n} \Pr \left( X_i \prod_{j=1}^{i} Y_j > x \right)
\]
\[
\sim \left( \sum_{i=1}^{n} \left( \mathbb{E} \left[ Y^{\alpha/2} \right] \right)^{i-1} \right) \nu(A) F(\sqrt{x}) .
\]
To derive a corresponding asymptotic lower bound, introduce
\[
\tau(x) = \inf \left\{ k \in \mathbb{N} : \sum_{i=1}^{k} X_i \prod_{j=1}^{i} Y_j > x \right\},
\]
which is a stopping time with respect to the natural filtration generated by $(X_i, Y_i, i \in \mathbb{N})$. Then
\[
\psi(x; n) = \sum_{k=1}^{n} \Pr (\tau(x) = k) .
\]
(20.15)
For arbitrarily fixed $\varepsilon \in (0, 1)$, we have
\[
\Pr (\tau(x) = k)
\]
\[
= \Pr \left( \max_{1 \leq l \leq k-1} \sum_{i=1}^{l} X_i \prod_{j=1}^{i} Y_j \leq x, \sum_{i=1}^{k} X_i \prod_{j=1}^{i} Y_j > x \right)
\]
Thus, it suffices to prove relation (20.12). In addition, by Lemma 1.7 of Vervaat (1979), both

\[ \sum_{1 \leq i \leq k-1} X_i \prod_{j=1}^{i} Y_j \leq x, \quad \sum_{i=1}^{k-1} X_i \prod_{j=1}^{i} Y_j > -\varepsilon x, \quad X_k \prod_{j=1}^{k} Y_j > (1 + \varepsilon)x \]

\[ \Pr \left( X_k \prod_{j=1}^{k} Y_j > (1 + \varepsilon)x \right) \]

\[ \geq \Pr \left( \max_{1 \leq i \leq k-1} \sum_{i=1}^{l} X_i \prod_{j=1}^{i} Y_j > x, \quad \sum_{i=1}^{k-1} X_i \prod_{j=1}^{i} Y_j > -\varepsilon x, \quad X_k \prod_{j=1}^{k} Y_j > (1 + \varepsilon)x \right) \]

\[ = \Pr \left( \sum_{i=1}^{k-1} X_i \prod_{j=1}^{i} Y_j > -\varepsilon x, \quad X_k \prod_{j=1}^{k} Y_j > (1 + \varepsilon)x \right) \]

\[ \sim \left( E \left[ Y^{\alpha/2} \right] \right)^{k-1} \nu(A) F \left( \sqrt{(1 + \varepsilon)x} \right) + o(1) \Pr (X_k Y_k > (1 + \varepsilon)x) \]

\[ \sim (1 + \varepsilon)^{-\alpha/2} \left( E \left[ Y^{\alpha/2} \right] \right)^{k-1} \nu(A) F \left( \sqrt{x} \right), \]

where in the second last step we applied relation (20.14) to deal with the first probability and applied Lemma 6.2 of Tang and Yuan (2012) to deal with the last two probabilities. It follows that

\[ \Pr \left( \tau(x) = k \right) \geq \left( E \left[ Y^{\alpha/2} \right] \right)^{k-1} \nu(A) F \left( \sqrt{x} \right). \]

Substituting this into (20.15) yields the desired asymptotic lower bound for \( \psi(x; n) \).

(b) Note that the uniformity over \( n \in \mathbb{N} \) of the asymptotics in (20.11) is

an immediate consequence of relations (20.11) and (20.12) since the sequence

\[ \sum_{i=1}^{n} \left( E \left[ Y^{\alpha/2} \right] \right)^{i-1}, \quad n \in \mathbb{N}, \]

starts with 1 and increases to \( (1 - E \left[ Y^{\alpha/2} \right])^{-1} \). Thus, it suffices to prove relation (20.12). In addition, by Lemma 1.7 of Vervaat (1979), both \( \sum_{i=1}^{\infty} X_i \prod_{j=1}^{i} Y_j \) and \( \sum_{i=1}^{\infty} X_i^+ \prod_{j=1}^{i} Y_j \) converge almost surely.

Since \( \psi(x; \infty) \geq \psi(x; n) \) for every \( n \in \mathbb{N} \), it follows from Theorem 20.3.1(a) that

\[ \psi(x; \infty) \geq \frac{\nu(A)}{1 - E \left[ Y^{\alpha/2} \right]} F \left( \sqrt{x} \right). \]

Now we derive the corresponding asymptotic upper bound for \( \psi(x; \infty) \). For arbitrarily fixed \( n \in \mathbb{N} \) and \( 0 < \varepsilon < 1 \), we derive

\[ \psi(x; \infty) \leq \Pr \left( \sum_{i=1}^{\infty} X_i^+ \prod_{j=1}^{i} Y_j > x \right) \]

\[ \leq \Pr \left( \sum_{i=1}^{n} X_i^+ \prod_{j=1}^{i} Y_j > (1 - \varepsilon)x \right) + \Pr \left( \sum_{i=n+1}^{\infty} X_i^+ \prod_{j=1}^{i} Y_j > \varepsilon x \right) \]

\[ = I_1(x, n) + I_2(x, n). \]
By Theorem 20.3.1(a),

\[ I_1(x, n) \sim (1 - \varepsilon)^{-\alpha/2} \left( \sum_{i=1}^{n} \left( E[Y^{\alpha/2}] \right)^{i-1} \right) \nu(A) F(\sqrt{x}) . \]

To deal with \( I_2(x, n) \), choose some \( 0 < \delta < 1 \) such that the function
\[
m(s) = E \left[ \left( \frac{Y}{1-\delta} \right)^s \right]
\]
is smaller than 1 in a neighborhood of \( s = \alpha/2 \). Then choose some large \( n \) such that
\[
\sum_{i=n+1}^{\infty} (1 - \delta)^i < 1
\]
and choose \( 0 < \alpha_1 < \alpha/2 < \alpha_2 < \alpha \) such that both \( m(\alpha_1) \) and \( m(\alpha_2) \) are strictly smaller than 1. We derive

\[
I_2(x, n) \leq \Pr \left( \sum_{i=n+1}^{\infty} X_i \prod_{j=1}^{i} Y_j > \varepsilon x \sum_{i=n+1}^{\infty} (1 - \delta)^i \right)
\]
\[
\leq \sum_{i=n+1}^{\infty} \Pr \left( X_i \prod_{j=1}^{i} Y_j > \varepsilon (1 - \delta)^i x \right)
\]
\[
= \sum_{i=n+1}^{\infty} \Pr \left( X_i Y_i \prod_{j=1}^{i-1} Y_j > \varepsilon (1 - \delta)^i x \right) .
\]

Notice that in each term on the right-hand side above, \( X, Y \) has a regularly varying tail with index \( -\alpha/2 \) and is independent of \( \prod_{j=1}^{i-1} Y_j \). By Lemma 3.2 of Hao et al. (2012), for some \( c > 0, x_0 > 0 \), and all \( x > x_0 \),

\[
\frac{I_2(x, n)}{\Pr(XY > \varepsilon (1 - \delta)x)} \leq c \sum_{i=n+1}^{\infty} E \left[ \left( \prod_{j=1}^{i-1} Y_j \right)^{\alpha_1} \left( \prod_{j=1}^{i-1} Y_j \right)^{\alpha_2} \left( \prod_{j=1}^{i-1} \frac{Y_j}{1-\delta} \right)^{\alpha_2} \right]
\]
\[
\leq c \sum_{i=n+1}^{\infty} \left[ m(\alpha_1)^{i-1} + m(\alpha_2)^{i-1} \right] .
\]

The right-hand side above converges and, hence, it tends to 0 as \( n \to \infty \). Thus, \( I_2(x, n) \) is negligible in comparison to \( I_1(x, n) \) for large \( n \). It follows that

\[
\psi(x; \infty) \leq \frac{\nu(A)}{1 - E[Y^{\alpha/2}]} F(\sqrt{x}) .
\]

This concludes the proof of Theorem 20.3.1.

\[ \square \]

**Acknowledgments**

This work was supported by an NSF grant (CMMI-1435864).
References


Interplay of Insurance and Financial Risks with Bivariate Regular Variation


