CAT BOND PRICING UNDER A PRODUCT PROBABILITY MEASURE
WITH POT RISK CHARACTERIZATION

BY

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ABSTRACT

Frequent large losses from recent catastrophes have caused great concerns among insurers/reinsurers, who then turn to seek mitigations of such catastrophe risks by issuing catastrophe (CAT) bonds and thereby transferring the risks to the bond market. Whereas, the pricing of CAT bonds remains a challenging task, mainly due to the facts that the CAT bond market is incomplete and that the pricing usually requires knowledge about the tail of the risks. In this paper, we propose a general pricing framework based on a product pricing measure, which combines a distorted probability measure that prices the catastrophe risks underlying the CAT bond with a risk-neutral probability measure that prices interest rate risk. We also demonstrate the use of the peaks over threshold (POT) method to uncover the tail risk. Finally, we conduct case studies using Mexico and California earthquake data to demonstrate the applicability of our pricing framework.

KEYWORDS

CAT bond, distortion, earthquake, extreme value theory, generalized Pareto distribution, peaks over threshold, pricing, product measure.

1. INTRODUCTION

Recent decades have witnessed an unprecedented surge in the frequency and severity of catastrophes. The three costliest catastrophes since 1980 each cost insurers about $30 billion or more, and the 2017 Hurricane Irma is estimated to cause an insured loss of up to $55 billion.1 Concerned with such catastrophic losses, insurers/reinsurers are constantly seeking solutions to catastrophe risk mitigation. While reinsurance has been widely used as a traditional solution, with its limited market capacity it can only digest a fraction of catastrophe

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risks. Insurance linked securities (ILS) as an alternative risk transfer (ART) solution have recently become popular.

Generally speaking, ILS are financial securities that have a payoff linked to insurance risks and are designed to provide additional funds for insurers/reinsurers to pay large claims when triggered. The ART mechanism using ILS helps insurers/reinsurers raise additional risk capital from the capital market and, owing to the much larger size of the capital market, greatly enhances their risk-bearing capacity. Currently, the most commonly used ILS is catastrophe (CAT) bonds, which behave similarly to plain vanilla bonds if not triggered, and otherwise use some or all of their principal to reimburse the bond sponsor for insurance claim costs. This paper is to discuss the pricing of CAT bonds.

The market of CAT bonds and ILS has been a remarkable success. Setting out a quick expansion in 2005 after Hurricane Katrina, its outstanding capital grew from $6.6 billion to $15.9 billion within 2 years. Although the trend was cooled down by the 2008 financial crisis, it regained momentum during recent years, with outstanding capital increasing steadily from $14.4 billion in 2011 to $37.8 billion in 2018. In fact, many recent deals ended up selling more than initially planned. The success of the market roots in both its supply and demand sides. On the supply side, insurers need them as a complement to reinsurance for catastrophe risk transfer. The recognition of contingent capital as eligible risk capital by regulation frameworks, such as the Solvency II Directive and the Swiss Solvency Test (SST), offers another incentive for insurers to utilize CAT bonds. On the demand side, CAT bonds are appealing to investors seeking to diversify their current portfolios, because the catastrophic events CAT bonds cover usually have a low correlation with the financial market (see, e.g., Cummins and Weiss (2009), Lane and Beckwith (2009), and Galeotti et al. (2013)). Their high yields are also attractive to investors, especially in the current low interest environment.

As the CAT bond market expands, pricing CAT bonds becomes increasingly important and is attracting much research attention. A main challenge of the pricing task arises from the incompleteness of the CAT bond market, which does not admit a unique risk-neutral pricing measure. Various approaches have been developed in the literature to address this challenge, yet they appear to be rather diverse, far from unified, and sometimes even contradicting with each other. Below we provide a brief review of the existing approaches:

**Zero risk premium.** A few early works on this topic simply treat CAT bonds as zero-beta securities and assume a zero risk premium for the underlying catastrophe risks; see Cummins and Geman (1995), Cox and Pedersen (2000), Cox et al. (2000), Lee and Yu (2002), and Ma and Ma (2013), among others. To justify the zero risk premium, many cite Merton (1976), who, in the context of option pricing under jump-diffusion stock prices, argues that the jump components of stock prices are likely caused only by company specific events and therefore are not correlated with the “market,” and as a result should bear no risk premium. A similar argument for pricing CAT bonds is that the underlying
catastrophe risks only have marginal influence on the overall economy and do not pose systematic risk to the “market,” and hence their risk premiums should be set to zero (see, e.g., Lee and Yu (2002)). Now that we must understand the “market” as the insurance market when pricing catastrophe risks, there is actually empirical evidence showing that such localized catastrophes may indeed have substantial and possibly systematic influence (see, e.g., Gürtler et al. (2016)). Thus, assuming zero risk premium may no longer be reasonable.

*Arbitrage pricing theory (APT).* Notwithstanding the incomplete market and the untradable underlying risk index, Vaugirard (2003) vindicates APT for CAT bonds based on the argument that continuous changes in the risk index can be mimicked by available instruments such as energy, power, and weather derivatives or contingent claims on certain commodities. See also Nowak and Romaniuk (2013) for an extension. Muermann (2008) applies APT to CAT derivatives on a compound Poisson catastrophic loss process, which, under a martingale pricing measure characterized by Delbaen and Haezendonck (1989) and Aase (1992), is still a compound Poisson process but with severity and frequency both modified by the corresponding market prices of risk. Jarrow (2010) too applies APT to pricing CAT bonds. Assuming that the market for the London Inter-Bank Offered Rate (LIBOR) and CAT bonds is arbitrage free (hence that a martingale pricing measure exists), he obtains a closed-form solution for valuing CAT bonds following the pricing methodology based on the reduced-form models used to price credit derivatives. Braun (2011) investigates the pricing of CAT swaps and proposes a two-stage contingent claims pricing approach.

*Probability transform.* This pricing approach dates back to Venter (1991), who discovers that various pricing frameworks in insurance and finance are established under a risk-adjusted probability distribution. Lane (2000) proposes a 3-parameter model to calculate risk premiums of CAT bonds by using the Cobb–Douglas function to link the probability of the first loss and the conditional expected loss. Wang (1996, 2000, 2002, 2004) introduces a class of probability transforms including in particular the Wang transform (see relation (3.3) below), and develops an insurance pricing framework using these transforms. See also Denneberg (1994) for more discussions on probability distortion. The Wang transform has been widely applied to price various kinds of ILS; see Lin and Cox (2005, 2008), Denuit et al. (2007), Chen and Cox (2009), and Chen and Cummins (2010), among many others.

*Econometric approach.* Although widely used in other fields, this approach is applied to value CAT bonds only recently. An econometric approach to CAT bond pricing first identifies the factors that determine the bond premium and then uses the factors to price the bond. Braun (2016) tests a number of hypotheses about the reliance of CAT bond spread on its rating class, reinsurance underwriting cycle, etc., and proposes an econometric model to price CAT bonds in the primary market. Gürtler et al. (2016) use an econometric approach to examine whether and how financial crises and natural catastrophes affect
CAT bond premiums and analyze which bond-specific and macroeconomic factors influence CAT bond premiums. Most recently, Stupfler and Yang (2018) apply an econometric approach to analyze the determining factors of the CAT bond premium.

*Indifference pricing.* This approach can be understood as an extension of the certainty equivalence principle to more general and possibly dynamic settings. It aims to find an agent’s bid and ask indifference prices according to his/her risk preference described by a utility function. The bid indifference price, for example, is the price at which he/she is indifferent, in terms of his/her expected utility of the optimal terminal wealth, between investing and not investing in this particular security. Under dynamic settings, the prices are usually obtained in an implicit form as solutions to partial differential equations. Since its first introduction by Hodges and Neuberger (1989), indifference pricing approach has been widely used in the literature of pricing in incomplete markets. See, for example, Egami and Young (2008), Barrieu and Loubergé (2009), and Leobacher and Ngare (2016) for recent applications to pricing CAT bonds and derivatives.

*Two-step valuation.* This approach was recently introduced and axiomatically characterized by Pelsser and Stadje (2014). For a security that involves both financial and insurance risks, under a dynamic setting Pelsser and Stadje (2014) aim to identify its price that is both market consistent and time consistent. The two-step approach combines financial pricing with actuarial valuation, by first valuating, based on an actuarial premium principle, the security conditional on the future development of the financial risks, and then taking an expectation of the value under a financial-risk adjusted probability measure. Moreover, Dhaene et al. (2017) also investigate the two-step valuation approach in an adapted version and show that their classes of fair valuations, hedge-based valuations, and two-step valuations are identical.

In this paper, we continue on the study of CAT bond pricing and aim at a general pricing framework. Our pricing measure is constructed, by virtue of the independence between the insured catastrophes and the financial market, to be a product measure of two easy-to-calibrate individual pricing measures, one being a distorted measure that prices the underlying catastrophe risks, and the other being a risk-neutral measure that captures the impact of the performance of the financial market on the bond price. In other words, this product pricing measure completely separates the two sources of randomness and enables an integrated pricing framework for both the catastrophe insurance risk and the interest rate risk involved in the CAT bond. We would like to also point out that we consider here a distortion of a probability measure, which is different from the distortion of a single distribution function often appearing in the literature. As a result, we do not have to restrict on CAT bonds that are defined on a single major loss event.

Our pricing framework can be regarded as a hybrid of the probability transform approach and the APT approach in the literature. To a certain extent it
can also be regarded as a two-step valuation in a dynamic setting with the actuarial valuation specified by a distortion risk measure. Owing to the generality of the pricing measure introduced, the idea behind this pricing framework is transformative to the pricing of essentially all ILS in the market.

Furthermore, we demonstrate the use of the peaks over threshold (POT) method in modeling the catastrophe risks involved. As CAT bonds are designed to cover high layers of insurance losses, their pricing also faces the challenge of quantitatively understanding the tail of the underlying catastrophe risks, which must be derived from noisy data in the tail area. Extreme Value Theory (EVT) offers an effective way to address the challenge, through the use of the block maxima (BM) method and the POT method. See Section 4 for brief descriptions of these two methods. In one of our case studies, we show that the POT method can be naturally applied to estimate the tail of the earthquake magnitude distribution.

The rest of this paper consists of five sections. Section 2 describes the mechanism of a CAT bond and quantifies its general terms. Section 3, which is the main part of the paper, proposes a general pricing framework using a product pricing measure. After collecting some highlights of EVT in Section 4, we conduct case studies for a Mexico earthquake bond and a California earthquake bond in Section 5. Finally, Section 6 concludes.

2. Modeling CAT Bonds

2.1. The mechanism

CAT bonds are issued by collateralized special purpose vehicles (SPVs), usually established offshore by sponsors who are insurers/reinsurers. The SPV receives premiums from the sponsor and provides reinsurance coverage in return. The premiums are usually paid to the bond investors as part of coupon payments, which typically also contain a floating portion. The floating portion is linked to a certain reference rate, usually the LIBOR, reflecting the return from the trust account where the principal is deposited. When the specified triggering event occurs, the principal and, hence, also the coupon payments will be reduced so that some funds can be sent to the sponsor as a reimbursement for the claims paid. See Cummins (2008) for more details.

According to Artemis, as of December 2018, among the over five hundred historical issues of CAT bonds, there have been fifty transactions that have caused a loss of principal to investors. It is noteworthy that although CAT bonds are designed to provide investors with a pure trade of insurance risk and to eliminate credit risk via collateral accounts, the flaws of the Total Return Swap (TRS) structure that prevailed before the 2008 financial crisis have caused principal losses for investors. In fact, four of the fifty principal losses are due to the failure of Lehmann Brothers who acted as the swap counterparty. Since then much improvement has been made on the collateral structure to further reduce counterparty default risk.
In the design of CAT bonds, of great importance is the choice of trigger. In general, triggers can be categorized into indemnity triggers and non-indemnity triggers. Indemnity triggers trigger the bond according to the sponsor’s actual loss due to the specified catastrophic events, while in contrast, non-indemnity triggers are based on other quantities chosen to reflect or approximate the actual loss. Typical non-indemnity triggers currently in use include industry loss triggers, parameter triggers, modeled loss triggers, and hybrid triggers. We refer the reader to Dubinsky and Laster (2003), Guy Carpenter & Company (2007), and Cummins (2008) for related discussions.

We conclude this subsection by showing a real example, the 2015 Acorn Re earthquake CAT bond, which has motivated one of our case studies. It is a $300 million bond issued by Acorn Re Ltd. in July 2015 with a maturity date of July 2018 that ended up paying the full principal to the investors. The bond has a parametric trigger and would be triggered if there were occurrences of earthquakes around the West Coast of the USA with magnitude over a certain threshold. Specifically, the covered region is divided into about 430 predetermined earthquake box locations, with each box having a rough size of one degree of longitude by one degree of latitude. The bond would be triggered if an earthquake in some box location occurs with a magnitude exceeding the minimum magnitude predetermined for that box and a depth not greater than 50 kilometers. For some earthquake box locations, the minimum magnitude is fixed to be 7.5, and a covered earthquake occurrence will wipe out the principal completely, while for some other earthquake box locations the minimum trigger levels are set progressively as 8.2, 8.5, 8.7, and 8.9, and an occurrence at these levels will wipe off 25%, 50%, 75%, and 100% of the principal accordingly. We shall use this bond as a prototype for our discussion of CAT bond pricing in Section 5.2.

2.2. The trigger process

Consider a general CAT bond with maturity date $T$ and principal/face value $K$. The bond makes annual coupon payments to investors at the end of each year until the maturity date $T$ or until the principal is completely wiped out, and makes a final redemption payment on the maturity date $T$ if the principal is not wiped out. The coupons are structured to contain two parts, a fixed part as premium paid to the bond investors for the reinsurance coverage, and a floating part equal to the return, at the LIBOR, on the bond sale proceeds that are deposited in a trust.

Suppose that the coupon payments and final redemption are linked to the occurrences of certain specified catastrophes. In this paper, we only consider natural catastrophes such as earthquakes, floods, droughts, hurricanes, and tsunamis, although, technically, CAT bonds can also be designed to be linked to man-made catastrophes such as financial crises, terrorist attacks, and cyber-attacks. If the bond is triggered, part or even all of the principal is liquidated
from the collateral to reimburse the sponsor’s insurance losses, and as a result both the fixed and floating coupons are reduced according to the amount of principal left.

We model the trigger to be a nonnegative, nondecreasing, and right-
continuous stochastic process \( Y = \{Y_t, t \geq 0\} \) defined on a filtered physical probability space \((\Omega^1, \mathcal{F}^1, \{\mathcal{F}_t^1\}, P^1)\). Specifically, we assume that \( Y \) is of the stochastic structure

\[
Y_t = f(X_1, X_2, \ldots, X_{N_t}), \quad t \geq 0, \tag{2.1}
\]

where \( \{X_n, n \in \mathbb{N}\} \) is a sequence of nonnegative random variables, called severities, \( \{N_t, t \geq 0\} \) is a counting process (i.e., an integer-valued, nonnegative, and nondecreasing stochastic process), and \( f \) is a component-wise nondecreasing functional. In case \( N_t = 0 \), the value of \( Y_t \) is understood as a constant \( f(0) \). We make \( f \) a general functional so as to allow for different designs of CAT bonds. In the context of earthquake CAT bonds, for example, the trigger \( Y \) can be designed to be:

(i) the aggregate amount of losses due to earthquakes, modeled by

\[
Y_t = \sum_{j=1}^{N_t} X_j, \quad t \geq 0,
\]

with \( N_t \) the number of earthquakes by time \( t \) and each \( X_j \) the individual loss amount due to the \( j \)th earthquake;

(ii) the maximum magnitude of earthquakes, modeled by

\[
Y_t = \max_{1 \leq j \leq N_t} X_j, \quad t \geq 0,
\]

with \( N_t \) the same as in (i) and each \( X_j \) the magnitude of the \( j \)th earthquake;

(iii) the number of major earthquakes, modeled by

\[
Y_t = \sum_{j=1}^{N_t} 1(X_j > y), \quad t \geq 0,
\]

with \( N_t \) and each \( X_j \) the same as in (ii), \( y \) a high threshold, and \( 1_\Delta \) the indicator of an event \( \Delta \), which is equal to 1 if \( \Delta \) occurs and to 0 otherwise.

A more sophisticated example is given in Section 5.2. As we see, the trigger \( Y \) can be made either indemnity based or non-indemnity based.

In what follows, we only consider a standard structure for \( Y \) in which, under \( P^1 \), the severities \( X_n, n \in \mathbb{N}, \) are independent, identically distributed (i.i.d.) copies of a generic random variable \( X \) distributed by \( F \) on \([0, \infty)\), and the counting process \( \{N_t, t \geq 0\} \) is a Poisson process with rate \( \lambda > 0 \) that is independent of \( \{X_n, n \in \mathbb{N}\} \). Under this standard structure, we use \( \mathcal{L}(f, F, \lambda) \) to symbolize the law of \( Y \) under \( P^1 \).
Suppose that there is an arbitrage-free financial market, defined on another filtered physical probability space \((\Omega^2, \mathcal{F}^2, \{\mathcal{F}_t^2\}, P^2)\). Apparently, the performance of this financial market has a significant impact on the CAT bond price.

Given the two physical probability spaces \((\Omega^1, \mathcal{F}^1, \{\mathcal{F}_t^1\}, P^1)\) and \((\Omega^2, \mathcal{F}^2, \{\mathcal{F}_t^2\}, P^2)\), introduce the product space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)\) with \(\Omega = \Omega^1 \times \Omega^2\), \(\mathcal{F} = \mathcal{F}^1 \times \mathcal{F}^2\) (the smallest sigma field containing \(A_1 \times A_2\) for all \(A_1 \in \mathcal{F}^1\) and \(A_2 \in \mathcal{F}^2\)), \(\mathcal{F}_t = \mathcal{F}^1_t \times \mathcal{F}^2_t\) for each fixed \(t \geq 0\), and \(P = P^1 \times P^2\). An implication of the use of \(P = P^1 \times P^2\) is that the two probability spaces are assumed to be independent, which is reasonable in view of the low correlation between the occurrences of natural catastrophes and the performance of the financial market.

For all random variables defined on one of the two spaces, we can easily redefine them in the product space \((\Omega, \mathcal{F})\). Precisely, for random variables \(Y^1(\omega_1)\) and \(Y^2(\omega_2)\) defined on the spaces \((\Omega^1, \mathcal{F}^1)\) and \((\Omega^2, \mathcal{F}^2)\), respectively, we can extend them to \(Y^1(\omega_1, \omega_2) = Y^1(\omega_1)1_{\Omega^1}(\omega_2)\) and \(Y^2(\omega_1, \omega_2) = 1_{\Omega^2}(\omega_1)Y^2(\omega_2)\), so that they are defined on the product space \((\Omega, \mathcal{F})\). Then it is easy to verify that \(Y^1(\omega_1, \omega_2)\) and \(Y^2(\omega_1, \omega_2)\) are independent of each other under the product measure \(P\). Actually, for two Borel sets \(B_1\) and \(B_2\), we have

\[
\begin{align*}
P(\{Y^1(\omega_1, \omega_2) \in B_1, \ Y^2(\omega_1, \omega_2) \in B_2\}) \\
= P(\{Y^1(\omega_1) \in B_1\} \times \Omega^2, \ \Omega^1 \times \{Y^2(\omega_2) \in B_2\}) \\
= P^1 \times P^2(\{Y^1(\omega_1) \in B_1\} \times \{Y^2(\omega_2) \in B_2\}) \\
= P^1(\{Y^1(\omega_1) \in B_1\})P^2(\{Y^2(\omega_2) \in B_2\}) \\
= P(\{Y^1(\omega_1, \omega_2) \in B_1\})P(\{Y^2(\omega_1, \omega_2) \in B_2\}).
\end{align*}
\]

See Section 5 of Cox and Pedersen (2000) for a similar discussion. Hereafter, we always tacitly follow this interpretation when we have to extend random variables defined on the two individual spaces to the product space.

The remaining principal of the CAT bond at time \(t\) depends on the development of the trigger process \(Y\) over \([0, t]\). To quantify this, we introduce a payoff function \(\Pi(\cdot): [0, \infty) \to [0, 1]\), nonincreasing and right-continuous with \(\Pi(0) = 1\), such that the remaining principal of the CAT bond at any time \(t \in [0, T]\) is equal to \(K\Pi(Y_t)\).

An implication of this payoff function is that the occurrence of a triggering catastrophe at time \(t\) wipes off an amount of

\[
K\Pi(Y_{t-0}) - K\Pi(Y_t)
\]

from the principal, where \(Y_{t-0}\) denotes the value of \(Y\) immediately before time \(t\), identical to \(Y_t\) if no triggering catastrophe at time \(t\). As we see, the payoff function \(\Pi(\cdot)\) stipulates a plan of allocating the principal between the
investors and the sponsor according to the development of the trigger $Y$ and, hence, it plays a central role in the CAT bond’s design.

Before the bond’s maturity, both the fixed and floating coupons may be reduced due to a reduction in bond principal. More precisely, let the fixed coupon rate be $R$ per year, let the floating coupon rate be $i_t$ per year over year $t$ (i.e., from time $t-1$ to time $t$) for $t=1, \ldots, T$, and let $r_t$ be the annualized instantaneous risk-free interest rate at $t$ for $t \geq 0$, where the stochastic processes $\{i_t, t=1, \ldots, T\}$ and $\{r_t, t \geq 0\}$ are defined on $(\Omega^2, \mathcal{F}^2)$. Therefore, the bond investors receive

$$K(R + i_t) \Pi(Y_{t-1})$$

on each coupon payment date $t=1, \ldots, T$ and receive the remaining principal $K \Pi(Y_T)$ on the maturity date $T$ if it is not completely wiped out.

Denote the time of principal wipeout by

$$\tau = \inf\{t \in \mathbb{R}_+: \Pi(Y_t) = 0\}, \quad (2.2)$$

where, as usual, we understand $\inf\emptyset$ as $\infty$. If the wipeout occurs between two coupon dates, then at time $\tau$ the bond investors are paid an accrued coupon that is proportional to the elapsed coupon period prior to the bond’s wipeout. Precisely, the accrued coupon is calculated as $K \vartheta$, with

$$\vartheta = (\tau - \lfloor \tau \rfloor)(R + i_{\lfloor \tau \rfloor}) \Pi(Y_{\lfloor \tau \rfloor}), \quad (2.3)$$

where $\lfloor \cdot \rfloor$ denotes the floor function.

3. A PRODUCT PRICING MEASURE

In the spirit of the fundamental theory of asset pricing (see, e.g., Delbaen and Schachermayer (2006) and Björk (2009)), we express the price of the CAT bond as the expectation, under a certain pricing measure, of the discounted values of cash flows, conditional on the available information about the development of the trigger and the performance of the financial market. Thus, with a pricing measure $Q$ to be determined, the price at time $t \in [0, T]$ of the CAT bond is calculated as

$$P_t = KE_t^Q \left[ \sum_{s=\lfloor \tau \rfloor+1}^{\lfloor \tau \rfloor \wedge T} D(t, s)(R + i_s) \Pi(Y_{s-1}) + D(t, \tau) \vartheta 1_{(\tau \leq T)} + D(t, T) \Pi(Y_T) 1_{(\tau > T)} \right], \quad (3.1)$$

where $\tau$ and $\vartheta$ are defined in (2.2) and (2.3), respectively. Here and throughout the paper, we follow the convention that a summation over an empty index set is 0, use $E_t^Q[\cdot] = E^Q[\cdot | \mathcal{F}_t^1 \times \mathcal{F}_t^2]$ to denote the expectation under $Q$ conditional on the available information up to time $t$, and use

$$D(t, s) = \exp \left\{ - \int_t^s r_u du \right\}$$
to denote the corresponding discount factor over the interval \([t, s]\). For the last term in (3.1), due to the right-continuity of both \(\Pi\) and \(Y\), it is easy to see that \(\Pi(Y_T)1_{(T>T)} = \Pi(Y_T)\). For this reason, we shall omit \(1_{(T>T)}\) in this term. Notice that (3.1) gives the full price with the accrued coupon included if \(t\) is not a coupon date. For \(t\) being exactly a coupon payment date, formula (3.1) gives the price immediately after the coupon payment is made.

An advantage of our consideration is that, as formula (3.1) shows, it enables us to price securities that contain not only a terminal payoff but also intermediate payments, which becomes crucially important for our purpose. Another advantage is that, like APT, our pricing formula expressed as the conditional expectation under a pricing measure \(Q\) is automatically time consistent, while time inconsistency is often an issue in many other pricing frameworks in incomplete markets.

It is challenging to determine the pricing measure \(Q\) in the pricing formula (3.1). Although for some CAT bonds with simple structures the current active ILS market may contain securities that replicate their payoffs, in general, a perfect replication remains hard to come by, as is argued by, for example, Cox et al. (2000). This means that, usually, CAT bonds cannot be simply priced in terms of assets already traded and priced in the market. In this section, we instead develop a hybrid pricing framework in which the probability transform approach is employed to price the underlying catastrophe risks and the APT approach is employed to price the interest rate risk. Note that, because the floating coupons are linked to a reference rate, CAT bonds are subject to minimum interest rate risk, but it is hard to argue that there is no interest rate risk at all. After all, the risk free rate used for discounting purpose and the reference rate used for the floating coupons do not always shift in parallel. Although interest rate risk has been a second-order issue for past deals, it is not clear whether this will always be the case going forward.

To reflect investors’ demand for catastrophe risk premium, we follow Wang (1996, 2000, 2002, 2004) to apply the idea of distortion. It is noteworthy that unlike the distortion widely used in the literature, which distorts a single distribution function only, we need to distort a probability measure that applies to the entire trigger process. In this paper, we refrain from a more advanced probabilistic treatment on this issue and simply demonstrate that it can be achieved by distorting the severity distribution in (2.1).

Specifically, let \(g : [0, 1] \rightarrow [0, 1]\) be a distortion function (i.e., a nondecreasing and right-continuous function with \(g(0) = 0\) and \(g(1) = 1\)) such that

\[
g(q) \leq q, \quad q \in [0, 1],
\]

and introduce a distorted severity function

\[
\tilde{F}(x) = g \circ F(x) = g(F(x)), \quad x \in \mathbb{R}.
\]

Then define a distorted probability measure \(Q^1\) on \((\Omega^1, \mathcal{F}^1)\) under which \(\{X_n, n \in \mathbb{N}\}\) is a sequence of i.i.d. random variables with common distorted
distribution $\tilde{F} = g \circ F$, while $\{N_t, t \geq 0\}$ is still a Poisson process with the same rate $\lambda > 0$ and independent of $\{X_n, n \in \mathbb{N}\}$; that is, the trigger process $Y$ follows the law $\mathcal{L}(f, \tilde{F}, \lambda)$ under $Q^1$. The existence of this distorted probability measure $Q^1$ can be rigorously justified by first identifying all finite-dimensional distributions of $Y$ and then applying Kolmogorov's extension theorem (see, e.g., Theorem 2.1.5 of Øksendal (2003)). The probability measure $Q^1$ will be used to price the catastrophe insurance risk.

The following proposition shows that, remarkably, $Y$ becomes more heavy tailed under the distorted probability measure $Q^1$ than under the original probability measure $P^1$ (i.e., the riskiness of $Y$ is amplified under $Q^1$).

**Proposition 3.1.** It holds for every $t \geq 0$ and $x \in \mathbb{R}$ that

$$Q^1(Y_t > x) \geq P^1(Y_t > x).$$

**Proof.** Define on $(\Omega^1, \mathcal{F}^1, P^1)$ a nonnegative random variable $\tilde{X}$ distributed by $\tilde{F} = g \circ F$. By condition (3.2), it holds that

$$P^1(\tilde{X} > x) = 1 - g(F(x)) = 1 - F(x) = P^1(X > x), \quad x \in \mathbb{R},$$

meaning that $\tilde{X}$ is stochastically not smaller than $X$ under $P^1$. On the probability space $(\Omega^1, \mathcal{F}^1, P^1)$, introduce $\{\tilde{X}_n, n \in \mathbb{N}\}$ to be a sequence of i.i.d. copies of $\tilde{X}$ and independent of $\{N_t, t \geq 0\}$. Then, for every $0 \leq t \leq T$ and $x \in \mathbb{R}$,

$$Q^1(Y_t > x) = Q^1(f(X_1, X_2, \ldots, X_{N_t}) > x) \geq P^1(Y_t > x),$$

where the second last step is due to the component-wise monotonicity of $f$ and the stochastic ordering between $\tilde{X}$ and $X$ under $P^1$. ■

As one of the most useful distortion functions, the Wang transform is defined by

$$g(q) = \Phi(\Phi^{-1}(q) - \kappa), \quad q \in [0, 1],$$

where $\kappa > 0$ is a parameter and $\Phi$ is the standard normal distribution. The Wang transform is appealing for its mathematical tractability and parsimony and, as summarized in Section 1, it has been widely applied to pricing insurance...
risks. Its single parameter $\kappa$ reflects the extent to which the distribution of a risk under $P^1$ needs to be distorted to be more skewed towards the right for the pricing purpose, and thus can be understood as a representation of the market price of catastrophe risk. It is usually estimated by calibrating the model to the price data from existing deals, and then used to price new deals.

So far we have obtained the pricing measure $Q^1$ for the catastrophe insurance risk. As mentioned before, the performance of the financial market, through the risk-free interest rate process $\{r_t, t \geq 0\}$ and the LIBOR process $\{i_t, t \in \mathbb{N}\}$, may have a significant impact on the CAT bond price. Such an impact should be properly reflected in the pricing formula. This can be achieved by identifying a risk-neutral pricing measure $Q^2$ for the arbitrage-free financial market according to the well-established APT to price the interest rate risk (see, e.g., Björk (2009)).

Finally, the independence assumption between the two markets suggests that we should use the product measure $Q^1 \times Q^2$ as the pricing measure for the CAT bond. Substituting $Q = Q^1 \times Q^2$ into (3.1) yields

$$
P_t = KE_t^{Q^1 \times Q^2} \left[ \sum_{s=|t|+1}^{\lfloor T \rfloor} D(t, s)(R + i_s) \Pi(Y_{s-1}) + D(t, \tau) \theta 1_{(\tau \leq T)} + D(t, T) \Pi(Y_T) \right],
$$

(3.4)

where $\tau$, $D(t, s)$, $R$, and $i_s$ are defined before. We expand formula (3.4) by conditioning on $\mathcal{F}_t^1 \times \mathcal{F}_t^2$ and factorizing the expectation $E_t^{Q^1 \times Q^2}$, yielding

$$
P_t = KE_t^{Q^1 \times Q^2} \left[ \sum_{s=|t|+1}^{\lfloor T \rfloor} \Pi(Y_{s-1}) E_t^{Q^1 \times Q^2} \left[ D(t, s)(R + i_s) | \mathcal{F}_t^1 \times \mathcal{F}_t^2 \right] \right] + KE_t^{Q^1 \times Q^2} \left[ (\tau - |\tau|) \Pi(Y_{|\tau|}) 1_{(\tau \leq T)} E_t^{Q^1 \times Q^2} \left[ D(t, \tau)(R + i_\tau) | \mathcal{F}_t^1 \times \mathcal{F}_t^2 \right] \right]$$

$$
+ KE_t^{Q^1 \times Q^2} \left[ D(t, T) \Pi(Y_T) \right] = KE_t^{Q^1} \left( \sum_{s=|t|+1}^{\lfloor T \rfloor} \Pi(Y_{s-1}) E_t^{Q^2} \left[ D(t, s)(R + i_s) \right] \right)$$

$$
+ KE_t^{Q^1} \left[ (\tau - |\tau|) \Pi(Y_{|\tau|}) 1_{(\tau \leq T)} E_t^{Q^2} \left[ D(t, \tau)(R + i_\tau) | \tau \right] \right]$$

$$
+ KE_t^{Q^1} \left[ \Pi(Y_T) \right] E_t^{Q^2} \left[ D(t, T) \right].
$$

(3.5)

Concerning the computation of this quantity, a usual idea is to assume that the underlying stochastic processes are Markovian, and then simplify the conditional expectations by the Markov property. Although the $E_t^{Q^i} [\cdot]$ terms in (3.5) usually require simulation to evaluate, the $E_t^{Q^i} [\cdot]$ terms may be explicitly available under some commonly used models for these interest rate and reference rate processes; see Appendix for more discussions.
In the literature, the pricing of catastrophe insurance risk is sometimes obtained by simply attaching a constant risk premium, say, $\delta$, to the risk-free interest rate used in the pricing formula so that the discount factor $D(t, s)$ is modified to

$$
\tilde{D}(t, s) = \exp \left\{ - \int_{t}^{s} (r_u + \delta) du \right\},
$$

regardless of the actual physical distribution of the underlying catastrophe insurance risk (see, e.g., Zimbidis et al. (2007) and Shao et al. (2015)). Under certain circumstances, such a pricing scheme may be questionable since the resulting risk premiums for catastrophes at different levels will become indistinguishable. Our idea of determining the risk premium is essentially different from this in that our pricing measure $Q^1$ is linked directly to the physical probability measure $P^1$. If the randomness of $Y$ fades out, then the CAT bond converges to a straight bond in the arbitrage-free financial market and consequently the price of the former should converge to the price of the latter. Our pricing formula (3.4) allows such a convergence, while attaching a constant risk premium as in (3.6) does not.

4. HIGHLIGHTS OF EVT

As CAT bonds are designed to cover high layers of insurance losses, their pricing crucially relies on proper modeling of the underlying catastrophe risks. To this end, it is natural to consider using EVT, which, in the univariate case, addresses two fundamental issues: the distribution of the maximum of risk realizations and the distribution of the exceedance of a risk over a high threshold. The solutions are given in terms of the corresponding limit distributions. Either one could play a material role in the pricing of CAT bonds, depending on the specific structure of the CAT bond under consideration. Moreover, in view of the fact that such limit theorems in EVT are established under very mild conditions, they are expected to produce relatively robust estimates for the underlying risk quantities and hence to help alleviate model misspecification.

The first issue is addressed by the classical Fisher–Tippett–Gnedenko theorem, which states that under some mild conditions the normalized maximum asymptotically follows a generalized extreme value (GEV) distribution. Precisely, let $F$ be the distribution function of a risk variable $X$, and let $M_n$ be the maximum of a sample of size $n$ from $F$. Say that $F$ belongs to the max-domain of attraction of a nondegenerate distribution $H$, written as $F \in \text{MDA}(H)$, if there are normalizing constants $c_n > 0$ and $d_n \in \mathbb{R}$ such that

$$
\frac{M_n - d_n}{c_n} \overset{d}{\to} H, \quad n \to \infty,
$$

available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/asb.2019.11
where \( \rightarrow^d \) denotes convergence in distribution. By the Fisher–Tippett–Gnedenko theorem, \( H \) must be a member of the family of GEV distributions whose standard form is

\[
H_\xi(x) = \exp\left\{-\left(1 + \xi x \right)^{-1/\xi}\right\}, \quad 1 + \xi x > 0,
\]

where \( (1 + \xi x)^{-1/\xi} \) is interpreted as \( e^{-x} \) for \( \xi = 0 \). This unifies the three extreme value types, Fréchet, Gumbel, and Weibull, and its parametric structure greatly facilitates statistical inference.

Now we show how to apply this theorem to approximate the distribution of the maximum of risks within a period of time. This is typically realized through the BM method, which entails splitting available data into blocks and fitting the observations of the maximum within each block to the GEV distribution. Specifically, given a sample of \( n \times m \) i.i.d. observations, we divide it into \( m \) blocks. By relations (4.1) and (4.2), it is reasonable to assume the common distribution of the i.i.d. BM \( M_{nj}, j = 1, \ldots, m \), to be

\[
H_{\xi, \mu, \sigma}(x) = \exp\left\{-\left(1 + \xi \frac{x - \mu}{\sigma}\right)^{-1/\xi}\right\}, \quad 1 + \xi \frac{x - \mu}{\sigma} > 0,
\]

which belongs to a three-parameter family with shape parameter \( \xi \in \mathbb{R} \), location parameter \( \mu \in \mathbb{R} \), and scale parameter \( \sigma > 0 \). These parameters can be estimated by implementing a standard maximum likelihood estimation (MLE) procedure. For a given sample size, the choice of the block size (and hence, the number of blocks) is a trade-off between the accuracy of the MLE and the approximation to the GEV distribution. Related discussions can be found in Section 5.1.4 of McNeil and Frey (2015). Such a method is particularly useful when, for example, the CAT bond considered is an earthquake bond whose trigger rests on the yearly maximum earthquake magnitude within the specified region.

The second issue is addressed by the Pickands–Balkema–de Haan theorem, which states that, for \( F \in \text{MDA}(H_\xi) \) with a finite or infinite upper endpoint \( x_F \), there exists a positive scale function \( a(\cdot) \) such that, for \( x > 0 \) and \( 1 + \xi x > 0 \),

\[
\lim_{y \uparrow x_F} P\left(\frac{X - y}{a(y)} > x \mid X > y\right) = (1 + \xi x)^{-1/\xi}.
\]

This means that the scaled excess over a high threshold \( y \) converges weakly to the generalized Pareto distribution (GPD), giving rise to the POT method of using the exceedances in the data to estimate the tail of a risk.

By incorporating a scale parameter the GPD above can be generalized to

\[
G_{\xi, \beta}(x) = 1 - \left(1 + \xi \frac{x}{\beta}\right)^{-1/\xi}, \quad \beta > 0,
\]
where \( x > 0 \) if \( \xi \geq 0 \) and \( 0 < x < -\beta/\xi \) if \( \xi < 0 \). For a random variable \( X \) following the GPD \( G_{\xi,\beta} \), it is easy to derive its mean excess function,

\[
e(y) = E[X - y | X > y] = \frac{\beta + \xi y}{1 - \xi}, \quad \beta + \xi y > 0,
\]

which is a linear function; see also Theorem 3.4.13(e) of Embrechts et al. (1997). This linearity allows us to use mean excess plots to check whether or not a GPD is a good fit to the data.

Details of these concepts and results are available in standard monographs in EVT such as Embrechts et al. (1997), Beirlant et al. (2004), and de Haan and Ferreira (2006). The normalizing constants \( c_n \) and \( d_n \) in (4.1) and the scale function \( a(\cdot) \) in (4.3) are all explicitly expressed. For applications of EVT to insurance and finance in general, we refer the reader to Embrechts et al. (1999), Bali (2007), Donnelly and Embrechts (2010), Kellner and Gatzert (2013), and McNeil et al. (2015), among others, while for applications of EVT to pricing CAT bonds in particular, we refer the reader to Zimbidis et al. (2007), Li et al. (2008), Chen and Cummins (2010), Shao et al. (2015), and Aviv (2018), among others.

5. Case studies

In this section, we present two case studies to demonstrate the application of our pricing formula. The first is a simple application to the 2006 CAT-Mex bond to show that our model can be calibrated to earthquake data, and the second uses a hypothetical earthquake bond, mimicking the 2015 Acorn Re bond, to show how EVT can be integrated into our pricing framework. We also use the second case to conduct sensitivity analysis of the price with respect to the level of earthquake risk and the market price of the risk.

5.1. A binary earthquake CAT bond

We study the larger tranche of the Mexico earthquake CAT bond issued by CAT-Mex Ltd. in May 2006, which is a binary bond covering Zone 2 of Mexico earthquake regions and would be triggered when a specified major earthquake hits this zone. We list details about the bond and our corresponding model in the example below.7

Example 5.1. Consider the 2006 CAT-Mex 3-year note, which has a principal of \( K = $150 \) million and coupons at an annual rate of 235 basis points over LIBOR payable quarterly. Its payoff function \( \Pi(\cdot) \) is defined by

\[
\Pi(y) = 1_{\{y = 0\}}, \quad y \geq 0,
\]

and trigger process \( Y_t \) given by the number of earthquakes during the 3-year term with epicenter in Zone 2 and magnitude 8.0 or greater.
To conform to (2.1), the process $Y_t$ defined above can be reasonably modeled as

$$Y_t = \sum_{j=1}^{N_t} \epsilon_j = \sum_{j=1}^{N_t} 1_{(\text{epicenter in Zone 2 and } X_j \geq 8)},$$

(5.1)

where $N_t$ is a homogeneous Poisson process with intensity rate $\lambda$ denoting the number of major earthquakes in Zone 2 by time $t$, and $X_j$ is the magnitude of the $j$th of the $N_t$ earthquakes. Without loss of generality, let $N_t$ be the number of earthquakes of magnitude 6.5 or greater. This facilitates our estimation process since our data set, described below, has complete records of such earthquakes.

Let each coupon period be $\Delta = 1/4$, and hence coupons be paid at times $s\Delta$, $s = 1, \ldots, 4T$. Then the accrued coupon when the bond is terminated before maturity becomes $K\vartheta$ with

$$\vartheta = (\tau - \lfloor 4\tau \rfloor \Delta) (R + i\tau),$$

and, according to relation (3.5), the time-0 price becomes

$$P_0 = \frac{K}{4} E^{Q^1} \left[ \sum_{s=1}^{\lfloor 4\tau \rfloor \wedge 4T} E^{Q^2}_t [D(0, s\Delta)(R + i\Delta s)] \right]$$

$$+ \frac{K}{4} E^{Q^1} \left[ (\tau - \lfloor 4\tau \rfloor \Delta) 1_{(\tau \leq T)} E^{Q^2}_t [D(0, \tau)(R + i\tau)] \right]$$

$$+ KQ^1(\tau > T) E^{Q^2}_t [D(0, T)].$$

(5.2)

Here time 0 is chosen as May 4, 2006, the pricing date of this bond.

5.1.1. Derivation of the pricing measure

To derive the pricing measure $Q^1$, we first estimate the earthquake risk parameters under $P^1$ and then find their corresponding values under $Q^1$. Denote by

$$p = P^1(\epsilon_i = 1)$$

the probability of a major earthquake, one with magnitude at least 6.5, triggering the bond. We use Mexico earthquake data from January 1900 to May 2006 obtained from National Seismological Service of National Autonomous University of Mexico to estimate the parameters $p$ and $\lambda$ under the $P^1$ measure. The data set has complete records of earthquakes of magnitude 6.5 or greater; see Figure 1 for a scatter plot of these records. We observe that there are 191 such earthquakes over the 100.25 years. Also, combining the data set and Table 1 of Härdle and Cabrera (2010), we see that among the 191 earthquakes, two had epicenter in Zone 2 and magnitude at least 8.0 and
Table 1

<table>
<thead>
<tr>
<th>Risk-free rate</th>
<th>LIBOR</th>
<th>Correlation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_r )</td>
<td>( b_r )</td>
<td>( \sigma_r )</td>
</tr>
<tr>
<td>0.140</td>
<td>5.62%</td>
<td>0.64%</td>
</tr>
</tbody>
</table>

Note: This table lists the parameters of the risk-free rate and LIBOR models, both assumed to be Vasicek, used for computing the bond price given by (5.2).

Figure 1: Scatter plot of Mexico earthquakes with magnitude 6.5 or greater.

Note: This figure is a scatter plot of Mexico earthquakes with magnitude 6.5 or greater during the period 1900–2006. Data source: National Seismological Service of National Autonomous University of Mexico in Mexico.

hence would trigger the bond. Therefore, we estimate the arrival intensity \( \lambda \) by \( \hat{\lambda} = 191/(100 + 1/4) = 1.905 \) and trigger probability \( p \) by \( \hat{p} = 2/191 = 0.010 \).

Under our pricing measure \( Q^1 \), the process \( Y_t \) is still a compound Poisson process with Poisson intensity \( \lambda \) and Bernoulli severity, of which the success (or, trigger) probability \( \tilde{p} = Q^1 (\epsilon_i = 1) \) is obtained through the Wang transform; that is,

\[
1 - \tilde{p} = \Phi(\Phi^{-1}(1 - p) - \kappa).
\]

We shall demonstrate that for a given level of the market price of risk \( \kappa \) one can use equation (5.2) to obtain the bond price, and then shall calibrate the \( \kappa \) such that at the offered coupon rate the issue price \( P_0 \) is equal to the face value of $150 million.

To determine the parameters used in (5.2) for the risk-free and LIBOR processes, we use 3-month U.S. treasury bill rates and 3-month U.S. dollar-based LIBORs and follow the procedure described in Appendix to estimate them; see Appendix for more details. We list the parameters in Table 1.
5.1.2. Numerical results
We first compute the bond price for evenly spaced values of $\kappa$ in $[0.5, 1]$. For every value, we find the bond price by a simulation of $10^5$ paths of a Poisson process with intensity $\hat{\lambda} \hat{p} = 0.02$ as the $Q^1$ trigger process $Y$. For each path, we first determine the arrival time of the first jump (i.e., the value of $\tau$) and then use relations (A.2) and (A.4) to evaluate the corresponding terms in (5.2). This allows us to obtain the prices according to (5.2), which are shown in Figure 2. Searching over the prices obtained we find that the $\kappa$ such that the bond is sold at par is $\kappa = 0.73$.

5.2. A per-occurrence-based earthquake CAT bond
Usually, more complex earthquake CAT bonds have payoffs linked to the maximum magnitude of earthquakes or the number of major ones during a period of time. For the former case, one may apply the BM method to approximate the distribution of the maximum magnitude, as described in Section 4. Note, however, that the BM method rests on the knowledge of the maxima of all blocks, which may not be available when, for example, only major earthquakes are recorded, as is exactly the case with our data. For the latter case, one may apply the POT method to characterize the tail of the earthquake magnitude. Here we construct a hypothetical earthquake CAT bond similar to the 2015 Acorn Re bond and discuss its pricing.
Example 5.2. Consider a T-year CAT bond with face value $K = 1000$, in which the payoff function $\Pi(y)$ is defined by

$$\Pi(y) = \max\{1 - y, 0\}, \quad y \geq 0,$$

and the trigger $Y$ is modeled by

$$Y_t = 0.25N_{1,t} + 0.5N_{2,t} + 0.75N_{3,t} + N_{4,t}, \quad t \geq 0, \quad (5.3)$$

with $N_{1,t}$, $N_{2,t}$, $N_{3,t}$, and $N_{4,t}$ denoting the numbers of earthquakes by time $t$ with magnitude in the ranges $(8.2, 8.5]$, $(8.5, 8.7]$, $(8.7, 8.9]$, and $(8.9, \infty)$, respectively, in a specified region. This means that the occurrence of every earthquake with sufficient magnitude will wipe off a fraction of the principal until possibly exhausting it.

To see that the trigger $Y$ defined by (5.3) possesses the stochastic structure of (2.1), we rewrite it as

$$Y_t = \sum_{j=1}^{N_t} \left(0.25 \times 1_{X_j \in (8.2,8.5]} + 0.5 \times 1_{X_j \in (8.5,8.7]} + 0.75 \times 1_{X_j \in (8.7,8.9]} + 1_{X_j \in (8.9,\infty]}\right),$$

where $N_t$ denotes the number of earthquakes by time $t$ and each $X_j$ for $j \in \mathbb{N}$ denotes the magnitude of the $j$th earthquake.

5.2.1. Data and estimation

Since it requires information about the tail of the earthquake distribution to price this bond, we use the earthquake catalog data of California\(^8\) provided by the California Department of Conservation’s California Geological Survey to estimate its distribution. The data list information about the earthquakes that occurred during 1769–2000 in California with a magnitude of at least 4.0, including the date and time of the occurrences, the latitudes and longitudes of their locations, and their magnitudes. In fact, the ones that occurred within 100 km of the state border and therefore could still cause damages to properties in California are also included. Although data are available for the year 1769 through 2000, only that after 1942 is complete. The full data set for the year 1769 through 2000 contains 5493 records in total. There are 4 records with a date of 0, which are for earthquakes that happened before 1840. We interpret the dates of 0 as records missing, and simply replace them by the first dates of the corresponding months. Also, we have deleted 7 records with a magnitude of 0. There may be multiple records of earthquake within a day for different locations across California. Since we are estimating the distribution of daily earthquake magnitude in California as a whole region, we only use the maximum record within a day. This leaves 3484 records in the data set.

From Figure 3, a scatter plot of the earthquake magnitudes, we see that there are significantly fewer records prior to the year 1942 than after, which is
a clear indication of data missing. Nonetheless, we may reasonably assume that the major earthquakes prior to the year 1942 are still recorded in the data. We should take into consideration such large records because they will likely have a noticeable impact on the tail of the fitted earthquake magnitude distribution and, hence, are crucially important for our pricing purpose. To locate the part of data prior to the year 1942 that can be deemed complete, we show in Figure 4 a comparison between the full data (1769–2000) and the complete data (1943–2000) in terms of the frequency of earthquakes with a magnitude above some level.

The left graph of Figure 4 shows the numbers of earthquakes per year with magnitude not smaller than \( x \), for \( 4.0 \leq x \leq 8.0 \), and the right graph is a zoomed-in version of the left one for \( 6.0 \leq x \leq 8.0 \). The numbers are shown for both the full data and the complete data. In view of the close frequencies of earthquakes starting from \( x = 7.0 \) for the two data sets, it is reasonable to assume that in the full data the records of earthquakes of magnitude over 7.0 are complete.

We first show some exploratory analysis based on the complete data with a scatter plot and a mean excess plot. In Figure 5, graph (a) is a scatter plot of the earthquakes with magnitude 4.0 or greater, and graph (b) is a mean excess plot, plotting the estimated mean excess function \( e(u) \), which, for every observation of earthquake magnitude, is estimated by the average excess over
FIGURE 4: Comparison of earthquake frequency: full data versus complete data.
NOTES: The left graph of the Figure 4 shows the numbers of earthquakes per year with magnitude not smaller than $x$, for $4.0 \leq x \leq 8.0$. The right graph is a zoomed-in version of the left one for $6.0 \leq x \leq 8.0$. In each graph, a comparison between the frequencies for the full data (1769–2000) and the complete data (1943–2000) is shown.

FIGURE 5: Scatter plot and mean excess plot.
NOTES: Graph (a) is a scatter plot of California earthquake magnitudes that are 4.0 or greater during the period 1943–2000. Graph (b) is the mean excess plot, also based on California earthquakes during 1943–2000. Graph (b) plots, for every observation of earthquake magnitude, the average excess over the magnitude observed in the data.

the magnitude. The mean excess plot offers important information regarding the tail of the earthquake magnitude distribution.

For comparison purpose, and to obtain a better fit in the tail area, we incorporate the large records prior to the year 1942 with magnitude over 7.0 and show a revised mean excess plot in Figure 6.

Recall from Section 4 that the mean excess plot of a GPD is linear. Therefore, Figure 5(b) suggests that it is reasonable to consider GPD while fitting the conditional distribution of the earthquake magnitude $X \mid X > y$ for some threshold $y$. Although the choice of threshold is a trade-off between bias...
and variance and is indeed subjective, it is suggested that we use a graphical approach to choose a threshold $y$ such that the empirical mean excess function is approximately linear beyond $y$. Therefore, a proper choice of the threshold can be $y = 5.0$; see Embrechts et al. (1999) for similar discussions. In the complete data (1943–2000) there are 334 records greater than 5.0, the exceedance probability $P(X > y)$ is roughly estimated as $334/(365 \times (2000 - 1943 + 1)) = 1.58\%$, and, hence, it is reasonable to consider $y = 5.0$ as a high threshold.

We then implement an MLE procedure to estimate the GPD parameters $\xi$ and $\beta$ based on the full data set (1769–2000) of earthquakes with magnitude 7 or greater and the complete data set (1943–2000) of earthquakes with magnitude between 5.0 and 7.0. Specifically, let $x_1, \ldots, x_m$ be the observations of all earthquake magnitudes over 7.0, let $x_{m+1}, \ldots, x_{m+n}$ be those between 5.0 and 7.0, and let $\tilde{X}_j = x_j - 5$ be the exceedance over the threshold 5.0, $j = 1, \ldots, m + n$. In our data set, we have $m = 18$ and $n = 328$.

We use the R function \texttt{optim} to maximize the (conditional) log-likelihood function

$$l(\xi, \beta | x_1, \ldots, x_{m+n}) = \frac{m}{\xi} \ln (\beta + 2\xi) - n \ln \left( \beta^{-1/\xi} - (\beta + 2\xi)^{-1/\xi} \right) - \frac{\xi + 1}{\xi} \sum_{j=1}^{m+n} \ln \left( \beta + \xi \tilde{X}_j \right).$$
with respect to $\xi$ and $\beta$, and obtain $\hat{\xi} = -0.127$ and $\hat{\beta} = 0.606$. The MLEs are known to have good properties such as consistency and asymptotic efficiency for $\xi > -1/2$ (see Section 6.5 of Embrechts et al. (1997)). The PP plot and QQ plot in Figure 7 show that the fitting is reasonably good. Nonetheless, we point out that the choice of the high threshold $y$ may have a significant impact on the estimation results, as can be seen from the graph of mean excess function. A smaller value of $y$ typically leads to estimates that correspond to a larger maximum possible magnitude of earthquake.

The estimates $\hat{\xi}$ and $\hat{\beta}$ indicate that the estimated maximum earthquake magnitude is 9.77. Note that the biggest earthquake ever recorded is the 1960 Valdivia earthquake in Chile with a magnitude of 9.5 (see Kanamori and Anderson (1975)).

5.2.2. The bond price and sensitivity analysis

Further, we make the following standard assumptions: Under $P^1$, the number of earthquakes in the region constitutes a Poisson process $\{N(t), t \geq 0\}$, and the magnitudes of individual earthquakes form a sequence of i.i.d. nonnegative random variables $\{X_j, j \in \mathbb{N}\}$ independent of $\{N(t), t \geq 0\}$. Hence, the insurance risk trigger $Y$ defined by (5.3) is a Markov process under $P^1$.

We then use the pricing formula (3.5) to price the bond and conduct numerical studies to demonstrate the impacts on the bond price of the parameter $\kappa$ of the Wang transform and the shape parameter $\xi$ of the earthquake magnitude distribution.

In our base model, we consider a bond with face value $1000, maturity $T = 1$ year, and fixed coupon rate 3.4% per year. The parameters we use for the two Vasicek models (A.1) and (A.3), as estimated in Appendix, and the parameters for the distribution of daily earthquake magnitude, $X$, under both $P^1$ and $Q^1$, are summarized in Table 2.
Table 2

SPECIFICATION OF INTEREST RATE PROCESSES AND EARTHQUAKE MAGNITUDE DISTRIBUTION
CASE STUDY IN SECTION 5.2.

<table>
<thead>
<tr>
<th>Panel A: Vasicek models (under $Q^2$)</th>
<th>Risk-free rate</th>
<th>LIBOR</th>
<th>Correlation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_r$</td>
<td>4.75</td>
<td>1.73</td>
<td>1.90</td>
</tr>
<tr>
<td>$b_r$</td>
<td>1.85%</td>
<td>2.91%</td>
<td>0.90</td>
</tr>
<tr>
<td>$\sigma_r$</td>
<td>0.18%</td>
<td>0.098%</td>
<td>0.2835%</td>
</tr>
<tr>
<td>$r_0$</td>
<td>0.1%</td>
<td>0.283%</td>
<td>0.098%</td>
</tr>
<tr>
<td>$\ell_0$</td>
<td>1.73</td>
<td>2.91%</td>
<td>0.098%</td>
</tr>
<tr>
<td>$\rho$</td>
<td>2.91%</td>
<td>0.098%</td>
<td>0.2835%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel B: Earthquake magnitude distribution</th>
<th>Under $P^1$</th>
<th>Under $Q^1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P^1(X &gt; 5) = 1.58%$</td>
<td>$Q^1(X \leq x) = \Phi(\Phi^{-1}(P^1(X \leq x)) - \kappa)$</td>
<td></td>
</tr>
<tr>
<td>$X - 5</td>
<td>(X &gt; 5) \sim GPD_{\xi, \beta}$</td>
<td>$\kappa = 1.23$</td>
</tr>
<tr>
<td>$\xi = -0.127$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\beta = 0.606$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Notes: This table lists the parameters of the interest rate models (Panel A) and the specification of daily earthquake magnitude distribution (Panel B) used in our numerical analysis. The two interest rate processes, risk-free rate and LIBOR, are assumed to follow the Vasicek models. The parameters are estimated using the procedure described in Appendix. For the earthquake magnitude distribution, it suffices to specify its tail when pricing CAT bonds. We list the specification under both the $P^1$ and $Q^1$ measures.

Under $P^1$, the daily earthquake magnitudes are i.i.d. with a 1.58% probability of exceeding 5.0, and conditional on its exceedance the amount of exceedance follows a GPD with shape parameter $\xi = -0.127$ and scale parameter $\beta = 0.606$. To obtain the distorted measure $Q^1$ for pricing, we use the Wang transform with $\kappa = 1.23$. The value of $\kappa$ is actually a calibrated value, based on an assumption that a bond with the same structure as ours is traded at par at issuance; see Figure 8 below. Such a calibration is motivated by the 2015 Acorn Re earthquake bond, which also has a fixed coupon rate of 3.4% per year and was indeed traded at par at issuance. Despite the differences, it is the closest to our example among all past issuances. The calibrated market price of catastrophe risk could be very different from those in the literature, since the distortion in our example is applied to the distribution of the earthquake magnitude, while those in the literature could be applied to, for example, the distribution of bond principal loss.

Each outer expectation $E_t^{Q^1} [\cdot]$ in relation (3.5) is estimated using a simulation of $10^5$ paths of the process $Y$ under $Q^1$ for each of the 360 days per year. The measure $Q^1$ is obtained by applying the Wang transform (3.3) with $\kappa = 1.23$ to the earthquake magnitude distribution. For each path, we determine the wipeout time $\tau$ and apply relations (A.2) and (A.4) to compute the $E_t^{Q^2} [\cdot]$ terms.

In our sensitivity analysis, the parameters $\xi$ and $\beta$ are always linked so that $\beta/\xi = -0.606/0.127$ and, hence, the upper endpoint of the earthquake magnitude distribution is fixed. The numerical results are demonstrated in Figures 8 and 9.
FIGURE 8: Change of the bond price with respect to $\kappa$.

NOTE: This figure shows, for a 1-year earthquake bond with face value $1000, fixed coupon 3.4\% per year, and structure described in Example 5.2, how the time-0 bond price responds to the change of the market price of earthquake risk $\kappa$. In this sensitivity analysis, the risk-free rate and the LIBOR are assumed to follow the Vasicek models with parameters listed in Panel A of Table 2. The tails of the daily earthquake magnitude distribution, under both the physical measure and the pricing measure, are specified via the parameters in Panel B of Table 2.

FIGURE 9: Change of the bond price with respect to $\xi$.

NOTE: This figure shows, for a 1-year earthquake bond with face value $1000, fixed coupon 3.4\% per year, and structure described in Example 5.2, how the time-0 bond price responds to the change of the shape parameter, $\xi$, of the fitted GPD. The tails of the daily earthquake magnitude distribution, under both the physical measure and the pricing measure, are specified via the GPD distribution. Its shape and scale parameters, $\xi$ and $\beta$, are linked so that $\beta/\xi = 4.772$, and hence the maximum earthquake magnitude under the model is 9.77. In this sensitivity analysis, the market price of earthquake risk $\kappa$ is chosen to be 1.23, and the risk-free rate and the LIBOR are assumed to follow the Vasicek models with parameters listed in Panel A of Table 2.
We observe that the Wang transform parameter $\kappa$ as an indicator of the earthquake risk premium has a significant impact on the CAT bond price (see Figure 8). A larger value of $\kappa$ means a higher risk premium required by the bond investors, and therefore a lower bond price. For example, an increase of $\kappa$ from 1.23 to 1.5 reduces the bond price substantially from $1000 to $914. Also, the shape parameter $\xi$, which controls the tail behavior of the earthquake magnitude distribution, has a significant impact (see Figure 9). A smaller value of $\xi$ (together with $\beta$ adjusted according to $\beta/\xi = -0.606/0.127$) means a higher probability assigned to the tail and, hence, leads to a lower bond price. For example, a decrease of $\xi$ from $-0.12$ to $-0.15$ reduces the price dramatically from $1,018$ to $896$. Both observations are consistent with our intuition.

Figure 10 shows a sample path of bond price evolution for the bond, now assumed to have maturity 3 years. To obtain the price curve, we first simulate a sample path for the interest rate process $r_t$ and for the LIBOR process $\ell_t$, and then envision a scenario where a major earthquake with magnitude between 8.2 and 8.5 occurs 10 months after the bond’s issuance. Of course, instead of simply assuming the time and strength of the earthquake occurrence, we could simulate them; this is only to obtain a more illustrative graph. We then find the bond price for each day during the 3-year term. We observe that the bond price increases as it gets closer to a coupon payment date, and drops after the payment by an amount that is equal to the coupon, and also drops dramatically, by about 25%, after the major earthquake.
6. Concluding remarks

In this paper, we propose a general approach for CAT bond pricing, which is an integration of the probability transform approach and the APT approach in the literature. The pricing measure is constructed as the product of a distorted probability measure and a risk-neutral probability measure to price the underlying catastrophe risks and interest rate risk, respectively. We use two case studies, one of the May 2006 Mexico earthquake CAT bond and the other of a hypothetical California earthquake bond, to demonstrate the applicability of the proposed pricing framework. The latter also shows how the POT method can be integrated into the pricing framework by characterizing the tail of the earthquake magnitude. An empirical study to test the performance of our model with cross-sectional data would be ideal, which we leave to future research.

We would like to point out that the insurance risk trigger used in the paper is driven by a compound Poisson process characterized by its frequency and severity, but our distorted probability measure results from a distortion of the severity only. Under certain circumstances, it may be more relevant to also apply distortion to the frequency, as is done by Muermann (2008) through a profoundly different way. More broadly, it would be interesting to construct a distorted probability measure by applying distortion directly to a general insurance risk trigger process that not necessarily takes the functional form studied here.

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Notes

3. For details, see http://www.artemis.bm/deal_directory/cat-bond-losses.html.
4. For details, see http://www.naic.org/capital_markets_archive/120504.htm.

5. Details can be found at http://www.artemis.bm/deal_directory/acorn-re-ltd-series-2015-1/.

6. As a referee kindly points out to us, substantially increased trading activities in the current secondary market of CAT bonds have actually made close replications of their payoffs possible. In addition, there are regular price quotes for industry loss warranty (ILW) contracts that reference the same type of risk. Hence, for some CAT bonds of a simple structure, it may be possible to use existing products (including in particular those in the secondary market) to replicate their payoffs, and therefore to construct an implied risk-neutral pricing measure.


REFERENCES


Our pricing framework involves a continuous-time risk-free interest rate process \( \{ r_t, t \geq 0 \} \) and a discrete-time LIBOR process \( \{ i_t, t \in \mathbb{N} \} \), both defined on \((\Omega^2, \mathcal{F}^2)\). In order to apply some well-established continuous-time interest rate...
models, we consider a continuous-time version of the LIBOR process \( \{ \ell_t, t \geq 0 \} \) and let \( i_t = e^{\ell_t} - 1 \) for \( t \in \mathbb{N} \) as an approximation. We model \( \{ r_t, t \geq 0 \} \) and \( \{ \ell_t, t \geq 0 \} \) as two correlated Ornstein–Uhlenbeck (OU) processes (i.e., Vasicek models) under the risk-neutral measure \( Q^2 \), which have been widely used to model interest rates. An important reason to choose the Vasicek model over, for example, the Cox–Ingersoll–Ross (CIR) model is that there are observations of negative rates in the data that we collect to fit the two processes, to be described below. See, for example, Cairns (2018) for more discussions on interest rate models.

Specifically, we suppose that under \( Q^2 \) the risk-free interest rate \( \{ r_t, t \geq 0 \} \) follows a Vasicek process,

\[
dr_t = ar_t (b_t - r_t) \, dt + \sigma_r dW_{r,t}, \quad t \geq 0, \tag{A.1}
\]

where \( a_r \), \( b_r \), and \( \sigma_r \) are positive numbers, with \( a_r \) corresponding to the rate of mean reversion, \( b_r \) the long-run mean, and \( \sigma_r \) the volatility, and \( W_{r,t} \) is a standard Brownian motion under \( Q^2 \). The OU process is clearly a time-homogeneous Markov process. Also, under such a process for the short rate, we have

\[
E^Q_t [D(t, s)] = A(t, s) e^{-B(t,s)r_t}, \quad s \geq t \geq 0, \tag{A.2}
\]

where

\[
B(t, s) = \frac{1 - e^{-a_r(s-t)}}{a_r},
\]

\[
A(t, s) = \exp \left\{ \frac{(B(t, s) - (s-t)) \left( a_r^2 b_r - \sigma_r^2 / 2 \right)}{a_r^2} - \frac{\sigma_r^2 B(t, s)^2}{4a_r} \right\}.
\]

Moreover, assume that under \( Q^2 \) the LIBOR \( \{ \ell_t, t \geq 0 \} \) follows another OU process,

\[
d\ell_t = a_\ell (b_\ell - \ell_t) \, dt + \sigma_\ell dW_{\ell,t}, \quad t \geq 0, \tag{A.3}
\]

where \( a_\ell \), \( b_\ell \), and \( \sigma_\ell \) are positive numbers interpreted similar to those in (A.1), and \( W_{\ell,t} \) is another standard Brownian motion under \( Q^2 \), satisfying

\[
dW_{r,t}dW_{\ell,t} = \rho dt, \quad t \geq 0,
\]

for some \( \rho \in (-1, 1) \).

Note that the bivariate stochastic process \( \{(r_t, \ell_t), t \geq 0\} \) is obviously a bivariate Markov process. Using a standard approach for affine term-structure models, we obtain that the term \( E^Q_t [D(t, s) i_t] \) in relation (3.5) is given by

\[
E^Q_t [D(t, s) i_t] = E^Q_t [D(t, s) e^{\ell_t}] - E^Q_t [D(t, s)]
\]

\[
= \tilde{A}(t, s) \exp \left\{ -B(t, s)r_t + \tilde{B}(t, s)\ell_t \right\} - A(t, s) e^{-B(t,s)r_t}, \quad s \geq t \geq 0, \tag{A.4}
\]
where \( A(\cdot, \cdot) \) and \( B(\cdot, \cdot) \) are defined as above,

\[
\tilde{B}(t, s) = e^{-a_t(s-t)}, \\
\tilde{A}(t, s) = \exp\left\{-(C_1(t, s) + C_2(t, s))\right\}
\]

with

\[
C_1(t, s) = \left(b_r - \frac{\sigma_r^2}{2a_r}\right)(s - t) + \frac{3\sigma_s^2}{4a_t} + \frac{\rho\sigma_s\sigma_t}{a_t(a_t - a_s)} + \frac{\sigma_t^2}{4a_t} + b_t - b_r,
\]

\[
C_2(t, s) = \frac{\sigma_r^2}{4a_t^2}e^{-2a_t(s-t)} + \left(\frac{b_r}{a_t} - \frac{\sigma_r^2}{a_t^2}\right)e^{-a_t(s-t)} + \left(\frac{\rho\sigma_s\sigma_t}{a_t(a_t - a_s)} - b_t\right)e^{a_t(s-t)} - \frac{\rho\sigma_s\sigma_t}{a_t(a_t - a_s)}e^{(a_t - a_s)(s-t)} - \frac{\sigma_t^2}{4a_t}e^{2a_t(s-t)}.
\]

**Proof of Equation (A.4).** Note again that \{(r_t, \ell_t)\}_{t \geq 0} is a bivariate Markov process. Hence,

\[
E_t^{Q^2} \left[ e^{-\int_t^T r_t dt + \ell_T} \right] = h(t, s, r_t, \ell_t), \quad s \geq t \geq 0,
\]

holds for some function \( h \). We guess that \( h \) is of the form

\[
h(t, s, r_t, \ell_t) = \tilde{A}(t, s) \exp\left\{-B(t, s)r_t + \tilde{B}(t, s)\ell_t\right\}, \quad s \geq t \geq 0, \tag{A.5}
\]

for some deterministic differentiable functions \( \tilde{A}, \tilde{B}, \) and \( B \). For brevity, we omit the arguments of these functions whenever no confusion could arise.

Setting \( E_t^{Q^2}[dh] = r_t h dt \) and using Itô’s lemma, we have, for any values of \( r_t \) and \( \ell_t \),

\[
\frac{\partial h}{\partial t} + a_r (b_r - r_t) \frac{\partial h}{\partial r} + a_\ell (b_\ell - \ell_t) \frac{\partial h}{\partial \ell} + \frac{1}{2} \sigma_r^2 \frac{\partial^2 h}{\partial r^2} + \rho\sigma_r\sigma_\ell \frac{\partial^2 h}{\partial r \partial \ell} + \frac{1}{2} \sigma_\ell^2 \frac{\partial^2 h}{\partial \ell^2} = r_t h. \tag{A.6}
\]

Using the expression of \( h \) given in (A.5), the left-hand side of (A.6) is equal to

\[
\frac{\partial \tilde{A}}{\partial t} e^{-Br_t + \tilde{B}\ell_t} + \left(\frac{\partial B}{\partial t} r_t + \frac{\partial \tilde{B}}{\partial t} \ell_t\right) h
\]

\[- a_r (b_r - r_t) B h + a_\ell (b_\ell - \ell_t) \tilde{B} h
\]

\[+ \frac{1}{2} \sigma_r^2 B^2 h - \rho\sigma_r\sigma_\ell B \tilde{B} h + \frac{1}{2} \sigma_\ell^2 B^2 h.\]
It follows that

\[- \frac{\partial B}{\partial t} + a_r B = 1, \]

\[\frac{\partial \tilde{B}}{\partial t} - a \ell \tilde{B} = 0,\]

\[\frac{1}{A} \frac{\partial \tilde{A}}{\partial t} - a \ell b R + a \ell b \ell \tilde{B} + \frac{1}{2} \sigma_r^2 B^2 - \rho \sigma_r \sigma_\ell B \tilde{B} + \frac{1}{2} \sigma_\ell^2 \tilde{B}^2 = 0.\]

The boundary condition \(h(s, s, r, \ell) = e^{\ell s}\) leads to

\[\tilde{A}(s, s) = 1, \quad B(s, s) = 0, \quad \tilde{B}(s, s) = 1.\]

Solving the system of ordinary differential equations above gives that

\[B(t, s) = \frac{1 - e^{-a_r(s-t)}}{a_r}, \]

\[\tilde{B}(t, s) = e^{-a_t(s-t)}, \]

\[\tilde{A}(t, s) = \exp\{-(C_1(t, s) + C_2(t, s))\},\]

with

\[C_1(t, s) = \left(b_r - \frac{\sigma_r^2}{2a_r}\right)(s-t) + \frac{3\sigma_r^2}{4a_r^2} + \frac{\rho \sigma_r \sigma_\ell}{a_\ell(a_\ell - a_r)} + \frac{\sigma_\ell^2}{4a_\ell} + b_\ell - \frac{b_r}{a_r},\]

\[C_2(t, s) = \frac{\sigma_r^2}{4a_r^2} e^{-2a_r(s-t)} + \left(\frac{b_r}{a_r} - \frac{\sigma_r^2}{a_r^2}\right) e^{-a_r(s-t)} + \left(\frac{\rho \sigma_r \sigma_\ell}{a_r a_\ell} - b_\ell\right) e^{a_\ell(s-t)} - \frac{\rho \sigma_r \sigma_\ell}{a_r(a_\ell - a_r)} e^{(a_\ell-a_\ell)(s-t)} - \frac{\sigma_\ell^2}{4a_\ell} e^{2a_\ell(s-t)}.\]

It is easy, though tedious, to verify that the function \(h\) given by (A.5), with \(\tilde{A}, B,\) and \(\tilde{B}\) specified above, satisfies the partial differential equation (A.6). Hence, by the Feynman–Kac theorem, \(h = E_t^{Q_2}\left[e^{-\int_t^s r(u)du + \ell u}\right]\). This, together with relation (A.2), completes the proof.

In our case studies, we follow the procedure described in Chapter 31 of Hull (2018) to estimate the parameters of the two Vasicek models for the risk-free rate and the LIBOR, using, respectively, 3-month U.S. treasury bill rates and 3-month U.S. dollar-based LIBORs. Specifically, for the binary bond in Section 5.1, we collect data for the period between January 2, 2000 and April 30, 2006, right before the issuance of the 2006 CAT-Mex bond, and for the per-occurrence-based bond, we collect data for the period between January 2, 2009 and July 2, 2015, right before the issuance of the 2015 Acorn Re bond.

For both cases, we first assume that the market price of interest rate risk is a constant, and hence the risk-free rate and LIBOR processes still follow
Vasicek models under $P^2$. Then we estimate the parameters of the two $P^2$ processes using either of the least square and maximum likelihood methods, which produce the same estimates. Finally, we estimate the market price of interest rate risk using quotes of treasury yields on May 4, 2006 and on July 2, 2015 (i.e., the pricing dates of the CAT-Mex bond and Acorn Re bond, respectively), with various durations (6 months to 30 years), and use the estimates to derive the parameters of the two interest rate processes under $Q^2$; see Hull (2018) for more details about the estimation procedure. The estimated parameter values are listed in Tables 1 and 2.