Multivariate Regular Variation in Insurance and Finance

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Contents

1. Multivariate Regular Variation
2. Losses Given Default
3. Capital Allocation
4. Interplay of Insurance and Financial Risks
5. Concluding Remarks
1. Multivariate Regular Variation
   1.1. Motivation
   1.2. Definition of MRV
   1.3. Special Case 1: a Linear Combination
   1.4. Special Case 2: a Mixture Model
   1.5. Special Case 3: the Archimedean Copula

2. Losses Given Default

3. Capital Allocation

4. Interplay of Insurance and Financial Risks

5. Concluding Remarks
In various situations of insurance, finance and risk management, we need to consider a portfolio consisting of $d$ losses, $X_1, \ldots, X_d$, leading to the aggregate amount

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It is in general much harder to model intangible dependence structures among them, which are, however, a main cause for contagion.
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The need to model dependence has become especially urgent today due to the complexity of insurance and financial products.
A positive measurable function $h$ on $[0, \infty)$ is said to be regularly varying at $\infty$ with regularity index $r \in (-\infty, \infty)$ if

$$\lim_{x \to \infty} \frac{h(xy)}{h(x)} = y^r, \quad y > 0.$$
Regular Variation

Definition

A positive measurable function $h$ on $[0, \infty)$ is said to be regularly varying at $\infty$ with regularity index $r \in (-\infty, \infty)$ if

$$
\lim_{x \to \infty} \frac{h(xy)}{h(x)} = y^r, \quad y > 0.
$$

- Bingham, Goldie, and Teugels (1987, Regular Variation)
- Resnick (1987, Extreme Values, Regular Variation, and Point Processes)
Multivariate Regular Variation (MRV)

Definition

A random vector \((X_1, \ldots, X_d)\) is said to have an MRV structure if there exists a non-degenerate limit measure \(\nu\) such that, for some distribution function \(F\) and every \(\nu\)-continuous set \(A \subset [0, \infty)^d\) away from \(0\),

\[
\lim_{x \to \infty} \frac{1}{F(x)} P \left( (X_1, \ldots, X_d) \in xA \right) = \nu(A).
\]

Resnick (2007, Heavy-Tail Phenomena)
Haan and Ferreira (2006, Extreme Value Theory: an Introduction)

The definition implies that the limit measure \(\nu\) is homogeneous: for some index \(0 < \alpha < \infty\),

\[
\nu(tB) = t^\alpha \nu(B)
\]

for all \(B \subset B\).

Hence, we write \((X_1, \ldots, X_d) \sim \text{MRV}_\alpha\).
A random vector \((X_1, \ldots, X_d)\) is said to have an **MRV structure** if there exists a non-degenerate limit measure \(\nu\) such that, for some distribution function \(F\) and every \(\nu\)-continuous set \(A \subset [0, \infty)^d\) away from 0,

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\lim_{x \to \infty} \frac{1}{F(x)} P \left( (X_1, \ldots, X_d) \in xA \right) = \nu (A).
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- Resnick (2007, *Heavy-Tail Phenomena*)
A random vector \((X_1, \ldots, X_d)\) is said to have an MRV structure if there exists a non-degenerate limit measure \(\nu\) such that, for some distribution function \(F\) and every \(\nu\)-continuous set \(A \subset [0, \infty)^d\) away from \(0\),

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The definition implies that the limit measure \(\nu\) is homogeneous: for some index \(0 \leq \alpha < \infty\),

\[
\nu(tB) = t^{-\alpha} \nu(B) \quad \text{for all } B \in \mathcal{B}.
\]

Hence, we write \(X \in \text{MRV}_{-\alpha}\).
Comments on the Limit Measure

The limit measure $\nu$ carries all asymptotic dependence information of $X$ in the upper-right tail:

$$\lim_{x \to \infty} \frac{1}{F(x)} P \left( \bigcap_{i=1}^{d} (X_i > x) \right) = \nu (1, \infty].$$
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Asymptotic dependence:

- $\nu(1, \infty] > 0$ means that $X^+$ exhibits large joint movements.
- If $\nu$ is concentrated on a straight line, then the components of $X^+$ are asymptotically fully dependent, for which comonotonicity and upper-comonotonicity are special cases.

References:

- Dhaene, Denuit, Goovaerts, Kaas, Vyncke (2002a, 2002b, IME)
- Cheung (2009, IME)
Comments on the Limit Measure

The limit measure $\nu$ carries all asymptotic dependence information of $\mathbf{X}$ in the upper-right tail:

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References:


Asymptotic independence:

- If $\nu(1, \infty] = 0$, then $\mathbf{X}^+$ does not exhibit large joint movements.
It follows easily from the definition of MRV that

\[
\lim_{x \to \infty} \frac{P(S_d > x)}{F(x)} = \nu(A)
\]

for \( A = \{(t_1, \ldots, t_d) : t_i \geq 0, \sum_{i=1}^{d} t_i > 1\} \).
The Sum under the MRV Structure

It follows easily from the definition of MRV that

$$ \lim_{x \to \infty} \frac{P(S_d > x)}{F(x)} = \nu(A) $$

for $A = \{(t_1, \ldots, t_d) : t_i \geq 0, \sum_{i=1}^d t_i > 1\}$.

**Implications:**

- Such a tail asymptotics can be immediately applied to compute tail-related risk measures such as VaR and CTE;
- In importance sampling, such a tail asymptotics can help identify a good importance sampling density.
  - Bassamboo et al. (2008, OR)
  - McLeish (2010, AB)
We model the $d$ dependent loss variables as

$$X = M\xi,$$

where $\xi = (\xi_1, \ldots, \xi_m)$ consists of i.i.d. inputs commonly distributed by $F$ and $M$ is an $d \times m$ random matrix independent of $\xi$. 
Special Case 1 - Assumptions

Assumption

- $F \in \mathcal{R}_{-\alpha}$ for some $\alpha \geq 0$
- $0 < E \|M\|^{\gamma} < \infty$ for some $\gamma > \alpha$
- $P \left( M_i M_j^\top > 0 \right) > 0$ for every $1 \leq i, j \leq d$, where $M_i$ and $M_j$ denote the $i$th and $j$th rows of $M$, respectively.

Trivially, this implies that $\xi \in \text{MRV}_m(-\alpha)$ with limit measure

$$\mu_\xi ([0, t]^c) = \sum_{i=1}^{m} t_i^{-\alpha}.$$
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$$\mu_{\xi} \left( [0, t]^c \right) = \sum_{i=1}^{m} t_i^{-\alpha}.$$

Corollary

The random vector $\mathbf{X}$ follows $\text{MRV}_d(-\alpha)$ with limit measure

$$\nu_1(\cdot) = E \left[ \mu_{\xi} \left( M^{-1} \cdot \right) \right].$$
Special Case 2: a Mixture Model

Now assume that $X$ follows a mixture structure

$$X = \sqrt{W} \left( \sqrt{\rho \eta_0} + \sqrt{1 - \rho \eta} \right).$$

- $W$: common shock
- $\eta_0$: systematic risk factor
- $\eta_1, \ldots, \eta_d$: idiosyncratic risk factors
- $\eta_0, \eta_1, \ldots, \eta_d$ are i.i.d. real-valued, independent of $W$
- $0 < \rho < 1$
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- $\eta_0, \eta_1, \ldots, \eta_d$ are i.i.d. real-valued, independent of $W$
- $0 < \rho < 1$

This model has been extensively applied to credit risk management:

- Frey and McNeil (2003, JR)
- Bassamboo, Juneja and Zeevi (2008, OR)
Special Case 2 - Result

Assumption

- $W$ has a regularly varying tail with index $-\alpha/2$ for some $\alpha \geq 0$
- $\eta_0, \eta_1, \ldots, \eta_d$ have finite moments of order $\gamma > \alpha$
Special Case 2 - Result

Assumption
- \( W \) has a regularly varying tail with index \(-\alpha/2\) for some \( \alpha \geq 0 \)
- \( \eta_0, \eta_1, \ldots, \eta_d \) have finite moments of order \( \gamma > \alpha \)

Corollary

The random vector \( \mathbf{X} \) follows \( \text{MRV}_d(-\alpha) \) with limit measure

\[
\nu_2(\cdot) = \mathbb{E} \left[ \mu \left( \mathbf{Z}^{-1} \cdot \right) \right],
\]

where \( \mu(\cdot) \) is defined by

\[
\mu \left( [\mathbf{0}, \mathbf{t}]^c \right) = \left( \bigwedge_{i=1}^{d} t_i \right)^{-\alpha}.
\]
Special Case 2 - Result

Assumption
- $W$ has a regularly varying tail with index $-\alpha/2$ for some $\alpha \geq 0$
- $\eta_0, \eta_1, \ldots, \eta_d$ have finite moments of order $\gamma > \alpha$

Corollary

*The random vector $X$ follows $\text{MRV}_d(-\alpha)$ with limit measure*

$$
\nu_2(\cdot) = E \left[ \mu \left( Z^{-1} \cdot \right) \right],
$$

where $\mu(\cdot)$ is defined by $\mu \left( [0, t]^c \right) = \left( \bigwedge_{i=1}^{d} t_i \right)^{-\alpha}$.

We can also consider the reverse case where $\eta_0, \eta_1, \ldots, \eta_d$ dominate $\sqrt{W}$. 
Special Case 3 - Assumptions

An Archimedean copula has the form

\[ C(u_1, \ldots, u_d) = \varphi^{-1}(\varphi(u_1) + \cdots + \varphi(u_d)). \]

- \( \varphi : [0, 1] \rightarrow [0, \infty] \), called the generator, is a strictly decreasing and convex function with \( \varphi(0^+) = \infty \) and \( \varphi(1-) = 0 \)
- \( \varphi^{-1} \) is the usual inverse of \( \varphi \)
An Archimedean copula has the form

$$C(u_1, \ldots, u_d) = \varphi^{-1}(\varphi(u_1) + \cdots + \varphi(u_d)).$$

- $\varphi : [0, 1] \rightarrow [0, \infty]$, called the generator, is a strictly decreasing and convex function with $\varphi(0+) = \infty$ and $\varphi(1-) = 0$
- $\varphi^{-1}$ is the usual inverse of $\varphi$

This expression indeed defines a proper copula if $\varphi^{-1}$ is completely monotone:

$$(-1)^k \frac{d^k}{dt^k} \varphi^{-1}(t) \geq 0, \quad k = 0, 1, \ldots.$$

- Nelsen (2006, *An Introduction to Copulas*)
Assumption

- \( X \) possesses the Archimedean copula above with equivalent regularly varying marginal tails
- There is some constant \( r \) such that

\[
\lim_{u \downarrow 0} \frac{\varphi(1 - tu)}{\varphi(1 - u)} = t^r, \quad t > 0.
\]

The second condition was proposed by Charpentier and Segers (2009, JMA).
Assumption

- **X** possesses the Archimedean copula above with equivalent regularly varying marginal tails
- There is some constant r such that

  \[
  \lim_{u \downarrow 0} \frac{\varphi(1 - tu)}{\varphi(1 - u)} = t^r, \quad t > 0.
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Corollary

*The random vector X follows MRV_d(−α) with limit measure \( \nu_3(\cdot) \) explicitly given.*
1. Multivariate Regular Variation

2. Losses Given Default
   2.1. Modeling Losses Given Default (LGD)
   2.2. The Main Result
   2.3. Numerical Results

3. Capital Allocation

4. Interplay of Insurance and Financial Risks

5. Concluding Remarks
Consider a portfolio of $d$ obligors (loans, bonds and other financial instruments subject to default).

Classical model for the LGD:

$$ L = \sum_{i=1}^{d} e_i \delta_i \mathbf{1}(X_i > a_i) $$

- $e_i$: positive deterministic exposure
- $\delta_i \in (0, 1)$: percentage loss
- $X_i$: loss variable of obligor $i$
- $a_i$: default threshold
The Classical Model

- Consider a portfolio of \( d \) obligors (loans, bonds and other financial instruments subject to default).
- Classical model for the LGD:

\[
L = \sum_{i=1}^{d} e_i \delta_i 1(X_i > a_i)
\]

- \( e_i \): positive deterministic exposure
- \( \delta_i \in (0, 1) \): percentage loss
- \( X_i \): loss variable of obligor \( i \)
- \( a_i \): default threshold

- This threshold model descends from Merton’s seminal firm-value work (Merton (1974, JF))
- The constant percentage loss ignores the severity of default
To take into account the severity of default, we introduce,

- severity of default: the percentage by which the loss variable exceeds the default threshold, that is,
  \[
  \left( \frac{X_i}{a_i} - 1 \right) +
  \]

- loss settlement function: for each \(i\), define a loss function \(G_i\) that is non-decreasing with \(G_i(y) = 0\) for \(y \leq 0\) and \(G(\infty) = 1\).
A New Model

To take into account the severity of default, we introduce,

- severity of default: the percentage by which the loss variable exceeds the default threshold, that is,
\[
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\]

- loss settlement function: for each $i$, define a loss function $G_i$ that is non-decreasing with $G_i(y) = 0$ for $y \leq 0$ and $G(\infty) = 1$.

The LGD:
\[
L = \sum_{i=1}^{d} e_i G_i \left( \frac{X_i}{a_i} - 1 \right),
\]

where the exposures are scaled such that $\sum_{i=1}^{d} e_i = 1$. 

Similar ideas of introducing a loss function can be found in the literature.
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- Pykhtin (2003, *RISK*) and Tasche (2004, *Working Paper*) employed such a loss function and even suggested the following specific form:

\[ G(y) = (1 - e^{-\mu - \sigma y})_+ . \]
Similar ideas of introducing a loss function can be found in the literature.

- Pykhtin (2003, *RISK*) and Tasche (2004, *Working Paper*) employed such a loss function and even suggested the following specific form:

  \[ G(y) = (1 - e^{-\mu - \sigma y})_+ . \]

- Schuermann (2004, *Credit Risk: Models and Management*) illustrated that the distribution of the recovery rate, and hence the loss function, contains two modes.
Our main focus is on the behavior of $L$ approaching its upper endpoint 1. Understand the upper endpoint 1 as the catastrophic edge.
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For this purpose, define

$$K = \frac{1}{1-L},$$

which is a continuous and strictly increasing function of $L$ and, hence, can be thought of as a proxy for $L$. 
The Main Result

Theorem (T. and Yuan (2013, NAAJ))

Assume that the random vector \( \{X_1^+, \ldots, X_d^+\} \) follows an MRV structure with limit measure \( \nu \). Upon some regularity conditions it holds that

\[
\lim_{x \to \infty} \frac{P(K > x)}{F(G(1 - 1/x))} = \tilde{\nu}(A),
\]

where

\[
A = \left\{ y \in [0, \infty)^d : \sum_{i=1}^d e_i / y_i < 1 \right\}
\]

and \( \tilde{\nu} \) is another limit measure mainly determined by \( \nu \).
Numerical Results

We conduct numerical studies to examine the accuracy:

- **Gumbel copula** with $1 < \beta < \infty$
- Losses $X_1$, $X_2$ and $X_3$ identically distributed by Pareto($\alpha$, $\theta$)
- **Loss settlement function**: $G(s) = 1 - (s + 1)^{-\gamma}$ for $s, \gamma > 0$.
- $(e_1, e_2, e_3) = (0.2, 0.4, 0.4)$
- $a_1 = a_2 = a_3 = a = 5$
- $c_1 = c_2 = c_3 = d_1 = d_2 = d_3 = 1$
- $\beta = 5$
- $\alpha = 2, \theta = 2$
- $\gamma = 0.8$

Marginal default probabilities calculated to be 0.082
Graph 7.1  Comparision of the estimates for the tail probability of K (N = 10,000,000)

![Graph showing comparison between simulated and asymptotic estimates for the tail probability of K with N = 10,000,000. The x-axis represents K, and the y-axis shows the probability Pr(K > x) multiplied by 1000. Two curves are plotted: one for simulated estimates and another for asymptotic estimates.](image-url)
Graph 7.2 Comparison of the estimates for the tail probability of K (N = 100,000,000)
**Graph 7.3** Comparision of the estimates for the VaR of K

![Graph](attachment://graph7.3.png)
Graph 7.4 Comparision of the estimates for the CTE of K

Simulated
Asymptotic

CTE_q[K]

q

Simulated/Asymptotic

q
1. Multivariate Regular Variation

2. Losses Given Default

3. Capital Allocation
   3.1. Euler’s Principle
   3.2. The Main Result
   3.3. Numerical Results

4. Interplay of Insurance and Financial Risks

5. Concluding Remarks
Consider an investor who invests in \( d \) different investment possibilities with loss-profit variables \( X_1, \ldots, X_d \). We consider how to allocate the risk capital \( \rho(S_d) \) to the individual investment possibilities such that

\[
\rho(S_d) = \sum_{i=1}^{d} AC_i,
\]

where \( \rho \) is a particular risk measure such as VaR or CTE.
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\[
\rho(S_d) = \sum_{i=1}^{d} AC_i,
\]

where \( \rho \) is a particular risk measure such as VaR or CTE.

Euler’s capital allocation principle says that if \( \rho \) is positively homogeneous and smooth, then

\[
AC_i = \left. \frac{\partial}{\partial \lambda_i} \rho \left( \sum_{i=1}^{d} \lambda_i X_i \right) \right|_{(1,\ldots,1)}.
\]
Under certain regularity conditions, it has been proven that

- For $\text{VaR}_q$,

  $AC_i = E [X_i | S_d = \text{VaR}_q[S_d]]$;

- For $\text{CTE}_q$,

  $AC_i = E [X_i | S_d > \text{VaR}_q[S_d]]$. 


As the excessive prudence of the current regulatory framework requires a confidence level close to 1, the notion of EVT becomes appropriate. We therefore study the asymptotic behavior of capital allocations as $q \uparrow 1$. 

\[ \lim_{q \to 1} AC_i F(q) = \frac{1}{\alpha} \nu(x: x_i > 1) + R \nu(x: x_j > z, \sum_{d j=1} x_j > 1). \]
The Main Result

As the excessive prudence of the current regulatory framework requires a confidence level close to 1, the notion of EVT becomes appropriate. We therefore study the asymptotic behavior of capital allocations as $q \uparrow 1$.

**Theorem (Asimit, Furman, T. and Vernic (2011, IME))**

Assume that $(X_1, \ldots, X_d)$ follows an MRV structure with a non-degenerate limit measure $\nu$. Upon some mild technical conditions, it holds that

$$
\lim_{q \uparrow 1} \frac{AC_i}{F^{-1}(q)} = \frac{\frac{1}{\alpha-1} \nu(x : x_i > 1) + \int_0^1 \nu \left( x : x_j > z, \sum_{j=1}^d x_j > 1 \right) dz}{\nu^{1-1/\alpha} \left( x : \sum_{j=1}^d x_j > 1 \right)}.
$$
Consider two risks $X_1$ and $X_2$, each following Pareto$(\theta, \alpha)$:

- $\theta$ equals 100,000 for $X_1$ and 150,000 for $X_2$
- $\alpha$ is the same for both risks and will be assigned values 2.5 and 3
Numerical Results

Consider two risks $X_1$ and $X_2$, each following Pareto($\theta, \alpha$):

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The dependence between $X_1$ and $X_2$ is given by the Gumbel copula.
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The dependence between \( X_1 \) and \( X_2 \) is given by the Gumbel copula.

Each analysis consists of 100 samples consisting of 5,000,000 simulations from \((X_1, X_2)\). The ratios between the capital allocations, estimated from the empirical distribution, and the approximation, are calculated for all 100 samples.
Table: Capital allocation ratio estimates for $\alpha = 2.5$

<table>
<thead>
<tr>
<th>$q$</th>
<th>$\beta = 2$</th>
<th>$\beta = 3$</th>
<th>$\beta = 5$</th>
<th>$\beta = 2$</th>
<th>$\beta = 3$</th>
<th>$\beta = 5$</th>
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</thead>
<tbody>
<tr>
<td>0.99</td>
<td>1.0705 (0.0059)</td>
<td>1.0733 (0.0045)</td>
<td>1.0748 (0.0045)</td>
<td>1.0730 (0.0053)</td>
<td>1.0747 (0.0049)</td>
<td>1.0751 (0.0042)</td>
</tr>
<tr>
<td>0.995</td>
<td>1.0500 (0.0074)</td>
<td>1.0536 (0.0067)</td>
<td>1.0548 (0.0066)</td>
<td>1.0522 (0.0076)</td>
<td>1.0543 (0.0067)</td>
<td>1.0551 (0.0065)</td>
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<tr>
<td>0.999</td>
<td>1.0231 (0.0127)</td>
<td>1.0243 (0.0130)</td>
<td>1.0263 (0.0126)</td>
<td>1.0261 (0.0136)</td>
<td>1.0263 (0.0130)</td>
<td>1.0264 (0.0129)</td>
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<td>1.0194 (0.0193)</td>
<td>1.0210 (0.0229)</td>
<td>1.0194 (0.0219)</td>
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### Table: Capital allocation ratio estimates for $\alpha = 3$

<table>
<thead>
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<th>$\beta = 3$</th>
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<tr>
<td>0.99</td>
<td>1.0826</td>
<td>1.0879</td>
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<td>(0.0040)</td>
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<tr>
<td>0.995</td>
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<td>(0.0097)</td>
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<td>(0.0143)</td>
<td>(0.0134)</td>
<td>(0.0134)</td>
</tr>
</tbody>
</table>
Contents

1. Multivariate Regular Variation

2. Losses Given Default

3. Capital Allocation

4. Interplay of Insurance and Financial Risks
   4.1. Insurance and Financial Risks
   4.2. The Complete Independence Case
   4.3. The Asymptotic Dependence Case

5. Concluding Remarks
Consider an insurance company who invests its wealth in a financial market consisting of riskless and risky assets.

Both insurance risk and financial risk may impair the solvency of the insurance company. A well-known question is which one of them plays a dominating role.
Consider an insurance company who invests its wealth in a financial market consisting of riskless and risky assets.

This company is exposed to the following two kinds of risk:

- **insurance risk**: the traditional liability risk, namely insurance claims, related to the insurance portfolio

- **financial risk**: the asset risk related to the investment portfolio including inflation and stock market crashes

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Consider the following discrete-time risk model:

- Denote by $X_n$ the insurer’s net loss (that is, the total amount of claims less premiums) within period $n$
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Consider the following discrete-time risk model:

- Denote by $X_n$ the insurer’s net loss (that is, the total amount of claims less premiums) within period $n$.
- Then the random variables $X_1, X_2, \ldots$ correspond to the insurance risk.
- Denote by $Y_n$ the stochastic discount factor (that is, $Y_n = (1 + R_n)^{-1}$ with $R_n$ the stochastic return rate) from time $n$ to time $n-1$.
- Then the random variables $Y_1, Y_2, \ldots$ correspond to the financial risk.
A Sum-Product Structure

The wealth of the insurance company accumulated till time $n$ is

$$W_n = x \prod_{j=1}^{n} (1 + R_j) + \sum_{i=1}^{n} (\text{Premium} - \text{Claim in Period } i) \prod_{j=i+1}^{n} (1 + R_j)$$

$$= \left( \prod_{j=1}^{n} (1 + R_j) \right) \left( x - \sum_{i=1}^{n} X_i \prod_{j=1}^{i} Y_j \right) = \left( \prod_{j=1}^{n} (1 + R_j) \right) (x - S_n).$$
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The sum-product structure

$$S_n = \sum_{i=1}^{n} X_i \prod_{j=1}^{i} Y_j$$

could be interpreted as the \textbf{stochastic present value} of aggregate net losses till time $n$. 
A Criticism on the Current Study

Assume that $(X_1, Y_1), (X_2, Y_2), \ldots$ are i.i.d. copies of $(X, Y)$. The mainstream of this study focuses on the case where:

- the insurance risk $X$ has a Pareto-type tail
- the financial risk $Y$ is dominated by the insurance risk $X$, that is, $G = o(F)$
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This dominating relationship truly holds if we consider the classical Black–Scholes market in which the log price is modeled as a Brownian motion.

- Grey (1994, AAP)
- Frolova, Kabanov and Pergamenshchikov (2002, FS)
- T. and Tsitsiashvili (2003, SPA)
- Paulsen (2008, PS)
However, empirical data often reveal that the lognormal model *significantly underestimates* the financial risk. It shows particularly poor performance in reflecting financial catastrophes such as the most recent one.
However, empirical data often reveal that the lognormal model significantly underestimates the financial risk. It shows particularly poor performance in reflecting financial catastrophes such as the most recent one. Such financial catastrophes intensify the need to investigate the opposite case where

- the financial risk $Y$ has a Pareto-type tail
- the insurance risk $X$ is dominated by the financial risk $Y$

For this case, the sum-product structure $S_n$ becomes much more intractable.
A distribution $V$ on $[0, \infty)$ is said to be \textit{convolution equivalent}, written as $V \in S(\alpha)$ for some $\alpha \geq 0$, if

$$
\lim_{x \to \infty} \frac{V(x - y)}{V(x)} = e^{\alpha y}, \quad y \in \mathbb{R},
$$

and

$$
\lim_{x \to \infty} \frac{V^{2*}(x)}{V(x)} = 2 \int_{-\infty}^{\infty} e^{\alpha x} V(dx) < \infty.
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For a distribution $U$ on $\mathbb{R}$, define

$$V(x) = 1 - \frac{U(e^x)}{U(0)}, \quad x \in \mathbb{R}.$$

**Definition**

A distribution $U$ is said to be of strongly regular variation if $V$ belongs to the class $S(\alpha)$ for some $\alpha \geq 0$. Write $U \in R_{-\alpha}^*$. 
Assumption

Every **convex combination** of $F$ and $G$ belongs to the class $\mathcal{R}^{*-\alpha}$, namely

$$pF + (1 - p)G \in \mathcal{R}^{*-\alpha}$$

for all $0 < p < 1$.

Essentially, we need strongly regular variation to guarantee the finiteness of the $\alpha$th moment.
The First Result

Assumption

Every **convex combination** of $F$ and $G$ belongs to the class $\mathcal{R}^{*}_{-\alpha}$, namely

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Essentially, we need strongly regular variation to guarantee the finiteness of the $\alpha$th moment.

Theorem (Li and T. (2014, $B$))

If $EY^{\alpha} < 1$ and $E \ln (X_\rightarrow \vee 1) < \infty$, then

$$P (S_{\infty} > x) \sim A_{\infty} \bar{F}(x) + C_{\infty} \bar{G}(x),$$

$$A_{\infty} = \frac{EY^{\alpha}}{1 - EY^{\alpha}} \quad \text{and} \quad C_{\infty} = \frac{ES_{\infty, +}^{\alpha}}{\mu_{\alpha}(1 - \mu_{\alpha})}.$$
If one of $\bar{F}$ and $\bar{G}$ dominates the other, then this term remains in the formulas but the other term vanishes. Otherwise, both terms should simultaneously appear in the formulas.

Thus, our result gives a unified answer to the question cited in the beginning of this talk.
If one of $\bar{F}$ and $\bar{G}$ dominates the other, then this term remains in the formulas but the other term vanishes. Otherwise, both terms should simultaneously appear in the formulas.

Thus, our result gives a **unified** answer to the question cited in the beginning of this talk.

**Computational issues**
If one of $\bar{F}$ and $\bar{G}$ dominates the other, then this term remains in the formulas but the other term vanishes. Otherwise, both terms should simultaneously appear in the formulas.

Thus, our result gives a unified answer to the question cited in the beginning of this talk.

Computational issues

An extension to the case that $(X, Y)$ follows a Farlie–Gumbel–Morgenstern distribution is immediate.
The Second Result

Assumption

- \((X, Y)\) follows MRV\(_{-\alpha}\) for \(\alpha > 0\)
- \(\nu (1, \infty] > 0\)
- \(\overline{G}(x) \sim c\overline{F}(x)\) for some \(c > 0\)
- \(0 < E[Y^{\alpha/2}] < 1\)

Hence, both risk variables are asymptotically dependent and tail equivalent.
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- \(0 < E[Y^{\alpha/2}] < 1\)

Hence, both risk variables are asymptotically dependent and tail equivalent.

Theorem (Yuan (2014, Working))

It holds for \(A = \{(x, y) \in [0, \infty]^2 : xy > 1\}\) that

\[
P(S_\infty > x) \sim \left(1 + \frac{E[Y^{\alpha/2}]}{1 - E[Y^{\alpha/2}]}ight) \nu(A) \overline{F}(\sqrt{x}).
\]
1. Multivariate Regular Variation

2. Losses Given Default

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4. Interplay of Insurance and Financial Risks

5. Concluding Remarks
I gave a brief introduction of multivariate regular variation. The limit measure carries all asymptotic dependence information in the upper-right tail.
Concluding Remarks

- I gave a brief introduction of multivariate regular variation. The limit measure carries all asymptotic dependence information in the upper-right tail.

- I illustrated its versatility in quantitative risk management by presenting its applications to various topics in the interdisciplinary area of insurance and finance.

Thank You Very Much!!!
I gave a brief introduction of multivariate regular variation. The limit measure carries all asymptotic dependence information in the upper-right tail.

I illustrated its versatility in quantitative risk management by presenting its applications to various topics in the interdisciplinary area of insurance and finance.

These topics include losses given default, capital allocation, and interplay of insurance and financial risks.
I gave a brief introduction of **multivariate regular variation**. The limit measure carries all **asymptotic dependence** information in the upper-right tail.

I illustrated its **versatility** in quantitative risk management by presenting its applications to various topics in the interdisciplinary area of insurance and finance.

These topics include **losses given default, capital allocation, and interplay of insurance and financial risks**.

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**Thank You Very Much!!!**