Extreme Risks in Insurance and Finance

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Outline

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   - The Max-Domain of Attraction
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The Black Swan
The Black Swan
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Natalie Portman in the 2010 movie
The Black Swan

Natalie Portman in the 2010 movie

A swimming black swan
Taleb (The Black Swan, 2007)

Before the discovery of Australia, people in the old world were convinced that all swans were white, an unassailable belief as it seemed completely confirmed by empirical evidence. The sighting of the first black swan...

Then he summarized the triplet of a Black Swan event:

- rarity
- extreme impact
- retrospective predictability

The prevalence of Black Swan events accompanied by disastrous economic and social consequences makes today’s world far different from just decades ago.
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The prevalence of Black–Swan events accompanied by disastrous economic and social consequences makes today’s world far different from just decades ago.
1992 Hurricane Andrew

- Total $16 billion in insured losses
- More than 60 insurance companies became insolvent, according to Muermann (2008, *NAAJ*)
- CBOT launched Insurance CAT futures contracts in 1992
9/11 Attacks

- Almost 3,000 died
- US stocks lost $1.4 trillion during the week
- By the end of 2002, New York City’s GDP estimated to have declined by $27.3 billion
2004 Indian Ocean Earthquake and Tsunami

- Damaged about $15 billion
- Over 230,000 were killed
- Not much insurance loss due to lack of insurance coverage
- A 2012 movie, *The Impossible*, based on the true story of a Spanish family
2005 Hurricane Katrina

- Damaged $108 billion, costliest one in the US
- Insured loss: $41.1 billion
2008 Sichuan Earthquake

- More than 90,000 died
- Damaged over $20 billion
- Insurers’ loss: 1 billion due to not much insurance coverage
2008 Recession

- Triggered by the collapse of the sub-prime mortgage market in the United States
- Arguably the worst global recession since the Great Depression in 1930’s
2010 Haiti Earthquake

- Killed more than 316,000 people
- Estimated cost: between $7.2--13.2 billion
2011 Japan Earthquake, Tsunami and Nuclear Crisis

- Deaths: Over 16,000
- Insured loss: $14.5-34.6 billion
- World Bank’s estimated economic cost: $235 billion
2012 Hurricane Sandy

- Damage: over $68 billion
- Insured loss: $19 billion
2013 Typhoon Haiyan/Yolanda

- Deaths: at least 6,241
- Missing: 1,785
- Damage: $1.5 billion
Central Limit Theorem (CLT)

- A sample of size $n$ distributed by $F$ with sum $S_n$

- CLT:

  $$\lim_{n \to \infty} P \left( \frac{S_n - b_n}{a_n} \leq x \right) = G_\alpha(x)$$

  for some constants $a_n > 0$ and $b_n \in \mathbb{R}$ and a non-degenerate distribution function $G_\alpha$ indexed by $\alpha \in (0, 2]$

- Write $F \in DA(G_\alpha)$

- $G_\alpha$ becomes the standard normal distribution when $\alpha = 2$
Extreme Value Theory (EVT)

- While focusing on extreme risks, we must employ EVT.

- Let $M_n$ be the maximum of the sample. EVT states:

$$\lim_{n \to \infty} P\left(\frac{M_n - d_n}{c_n} \leq x\right) = H_\xi(x)$$

for some constants $c_n > 0$ and $d_n \in \mathbb{R}$ and a non-degenerate distribution function $H_\xi$.

- Write $F \in \text{MDA}(H_\xi)$.
The Fisher–Tippett–Gnedenko Theorem (1927, 1948)

The limit distribution $H_\xi$ is a member of the generalized extreme value family of distributions with standard version

$$H_\xi(x) = \exp \left\{ -(1 + \xi x)^{-1/\xi} \right\}.$$
Three Extreme Value Distributions

According to the value of $\xi$, we have the following three cases:

- **Weibull** case: $\xi < 0$
- **Gumbel** case: $\xi = 0$
- **Fréchet** case: $\xi > 0$
Three Extreme Value Distributions

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Recent References on EVT


Peaks over Threshold (POT)

For $F \in \text{MDA}(H_\xi)$ with an upper endpoint $x_F$, there exists a positive function $a(\cdot)$ such that

$$\lim_{y \uparrow x_F} P \left( \frac{X - y}{a(1/F(y))} > x \left| X > y \right. \right) = (1 + \xi x)^{-1/\xi}. \quad (1)$$

The scaled excess over a high threshold converges weakly to a random variable $Y$ following the generalized Pareto distribution (GPD).
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Wedding on July 2nd, 2014
Haezendonck and Goovaerts in 1990s

Jean M. Haezendonck

Marc Goovaerts
Definition of the Haezendonck–Goovaerts Risk Measure

Let $X$ be a risk variable distributed by $F$. Let $\varphi$ represent a Young function, which is a nonnegative and convex function on $[0, \infty)$ with $\varphi(0) = 0$, $\varphi(1) = 1$ and $\varphi(\infty) = \infty$.

For $q \in (0, 1)$, the Haezendonck–Goovaerts (HG) risk measure is defined as

$$H_q[X] = \inf_{x \in \mathbb{R}} (x + h),$$

where $h$ solves the equation

$$\mathbb{E} \left[ \varphi \left( \frac{(X - x)_+}{h} \right) \right] = 1 - q.$$
Basic Properties

- Law invariant
- Coherent
- Reduces to CTE when $\varphi(s) = s$
References

- Haezendonck and Goovaerts (1982, *IME*) – Swiss premium calculation principle induced by the Orlicz norm
- Krätschmer and Zähle (2011, *IME*)
- Goovaerts, Linders, Van Weert and Tank (2012, *IME*)
- Mao and Hu (2012, *IME*)
- Ahn and Shyamalkumar (2014, *IME*)
- ...
A General Result

Under certain mild regularity conditions, the HG risk measure is equal to

$$H_q[X] = x_* + h_*,$$

where the pair \((x_*, h_*)\) solves

$$\begin{align*}
\mathbb{E}\left[\varphi\left(\frac{(X-x)_+}{h}\right)\right] &= 1 - q, \\
\mathbb{E}\left[\varphi'\left(\frac{(X-x)_+}{h}\right)\right] &= \mathbb{E}\left[\varphi'\left(\frac{(X-x)_+}{h}\right) \frac{(X-x)_+}{h} \right].
\end{align*}$$
An EVT Solution

Now assume $F \in \text{MDA}(H_\xi)$. Recall the GPD random variable $Y$ in (1) and write

$$\frac{(X - x)_+}{h} | (X > x) = \frac{a(1/\overline{F}(x))}{h} \cdot \frac{(X - x)_+}{a(1/\overline{F}(x))} | (X > x) \xrightarrow{d} kY,$$

where $k > 0$ is the unique solution of the equation

$$E \left[ \varphi'(kY) \right] = E \left[ \varphi'(kY) kY \right].$$
An EVT Solution

Now assume $F \in \text{MDA}(H_\xi)$. Recall the GPD random variable $Y$ in (1) and write

$$
\frac{(X - x)_+}{h} \mathbb{1}(X > x) = \frac{a(1/F(x))}{h} \cdot \frac{(X - x)_+}{a(1/F(x))} \mathbb{1}(X > x) \xrightarrow{d} kY,
$$

where $k > 0$ is the unique solution of the equation

$$
E \left[ \varphi'(kY) \right] = E \left[ \varphi'(kY) kY \right].
$$

Theorem 2.1

Under certain mild regularity conditions, as $q \uparrow 1$ we have

$$H_q[X] = x_* + h_*,$$

where the pair $(x_*, h_*)$ satisfies

$$
\bar{F}(x_*) \sim \frac{1 - q}{E[\varphi(kY)]} \quad \text{and} \quad h_* \sim \frac{a(1/F(x_*))}{k}.
$$
Application 1: The Haezendonck–Goovaerts Risk Measure

**Numerical Study: the Fréchet Case**

- Young function chosen as
  \[ \varphi(s) = \frac{s^{2.2} + s^{1.1}}{2} \]
- \( F \) is Pareto(\( \alpha \), 1)
Numerical Study: the Gumbel Case

- Young function chosen as
  \[ \varphi(s) = \frac{s^{2.2} + s^{1.1}}{2} \]
- \( F \) is lognormal(\( \mu = 2, \sigma = 0.5 \))
Young function chosen as

\[ \varphi(s) = \frac{s^{2.2} + s^{1.1}}{2} \]

\[ F \text{ is beta}(a = 2, b) \]
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6. Concluding Remarks
Consider an insurance portfolio consisting of $d$ losses, $X_1, \ldots, X_d$. The aggregate loss is

$$S_d = \sum_{i=1}^{d} X_i.$$ 

This can also be interpreted as the aggregate loss of an investor who invests in a fixed set of $d$ different investment possibilities.
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The loss variables should be mutually dependent because they are exposed to common or similar risk factors. The intricate dependence structure is a main cause for contagion.
A General Framework

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Question: How to model the dependence among the $d$ loss variables?
Multivariate Normal Model

Multivariate normal distributions fail to capture the following features of data in insurance and finance:

- pronounced asymmetry
- heavy-tailedness
- asymptotic dependence
Multivariate Normal Model

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- asymptotic dependence

Salmon (2012, *The formula that killed Wall Street*): Here’s what killed your 401(k) – David X. Li’s Gaussian copula:

\[
P(T_A < 1, T_B < 1) = \Phi_2(\Phi^{-1}(F_A(1)), \Phi^{-1}(F_B(1)), \gamma)
\]
Scatter Plots of Gaussian Copulas

\[ \rho = 0.1 \]

\[ \rho = 0.3 \]

\[ \rho = 0.7 \]

\[ \rho = 0.9 \]
**t-Copula**

A $t$-copula with degree of freedom $\nu$ and correlation matrix $P = (\rho_{ij})$ is given by

$$C_{\nu,P}(u_1, \ldots, u_d) = t_{\nu,P}(t_{\nu}^{-1}(u_1), \ldots, t_{\nu}^{-1}(u_d)),$$

- $t_{\nu,P}$: a $d$-dimensional $t$ distribution with degree of freedom $\nu$, mean 0, and positive definite dispersion matrix $P$,
- $t_{\nu}^{-1}$: the quantile function of a standard univariate $t$ distribution with degree of freedom $\nu$. 
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For every pair $(X_i, X_j)$, the coefficients of upper and lower tail dependence coincide and are equal to

$$\lambda_{ij} = 2t_{\nu+1} \left( -\sqrt{\frac{(\nu + 1)(1 - \rho_{ij})}{1 + \rho_{ij}}} \right) > 0.$$
Scatter Plots of $t$-Copula

$v = 5, \rho = 0.3$

$v = 4, \rho = 0.6$

$v = 3, \rho = 0.9$

$v = 2, \rho = 0.95$
Gumbel Copula

A Gumbel copula with parameter $\beta > 1$ is given by

$$C(u_1, \ldots, u_d) = \exp \left\{ - \left( \sum_{i=1}^{d} (-\ln u_i)^\beta \right)^{1/\beta} \right\}.$$
A Gumbel copula with parameter $\beta > 1$ is given by

$$C(u_1, \ldots, u_d) = \exp \left\{ - \left( \sum_{i=1}^{d} (-\ln u_i)^\beta \right)^{1/\beta} \right\}.$$ 

- The pairwise coefficient of lower tail dependence is 0;
- The pairwise coefficient of upper tail dependence is $2 - 2^{1/\beta} > 0$;
- The strength of dependence varies from independence to comonotonicity as $\beta$ varies from 1 to $\infty$. 

Qihe Tang (University of Iowa)
Scatter Plots of Gumbel Copula

\( \beta = 1 \)

\( \beta = 2 \)

\( \beta = 3 \)

\( \beta = 6 \)
Varying Strength of Dependence

Assume that the $d$ losses are affected by a risk factor $I$:

- **earthquake insurance** – the magnitude of an earthquake
- **agricultural insurance** – weather indices
- **health insurance** – the spread of a pandemic, the effect of a vaccine
- **banking and finance** – macroeconomic factors
Varying Strength of Dependence

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George E. P. Box (10/18/1919–03/28/2013): “… essentially, all models are wrong, but some are useful …”
Another useful framework is a mixture structure

\[ X_i = u_i + \sqrt{W} \left( \sqrt{\rho \eta_0} + \sqrt{1 - \rho \eta_i} \right), \quad i = 1, \ldots, d, \]

- \( u_i \in \mathbb{R} \) and \( 0 < \rho < 1 \) are nonrandom
- \( \eta_0 \): systematic risk factor
- \( \eta_1, \ldots, \eta_d \): idiosyncratic risk factors
- \( W \): a nonnegative random variable interpreted as common shock
- \( W, \eta_0, \eta_1, \ldots, \eta_d \) are independent
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This model has been extensively applied to credit risk management:

- Brereton et al. (2013, *Credit Portfolio Securitizations and Derivatives*)
- Bassamboo et al. (2008, *OR*)
- Frey and McNeil (2003, *JR*)
Multivariate Regular Variation (MRV)

All of the dependence structures mentioned above, equipped with Fréchet marginal distributions, can be unified to the MRV framework.
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A random vector \((X_1, \ldots, X_d)\) is said to have an MRV structure if there exists a non-degenerate limit measure \(\nu\) such that, for some distribution function \(F\) and every \(\nu\)-continuous set \(A \subset [0, \infty)^d\) away from \(0\),

\[
\lim_{x \to \infty} \frac{1}{F(x)} P((X_1, \ldots, X_d) \in xA) = \nu(A).
\]
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Reference:
The Sum under the MRV Structure

It follows easily from the definition of MRV that

$$\lim_{x \to \infty} \frac{P(S_d > x)}{F(x)} = \nu(A)$$

for $A = \{ (t_1, \ldots, t_d) : \sum_{i=1}^d t_i > 1 \}$. 
The Sum under the MRV Structure

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\[ \lim_{x \to \infty} \frac{P(S_d > x)}{F(x)} = \nu(A) \]

for \( A = \{(t_1, \ldots, t_d) : \sum_{i=1}^{d} t_i > 1 \} \).

Implications:

- Such a tail asymptotics can be immediately applied to compute tail-related risk measures such as VaR and CTE;
- In importance sampling, such a tail asymptotics can help identify a good importance sampling density. See Bassamboo et al. (2008, OR) and McLeish (2010, AB).
Great Generality for Modeling Extreme Dependence

Dependence in the tail of particular strength can be captured through a particular limit measure $\nu$.

**For example:**
- asymptotic independence
- asymptotic dependence
- asymptotic full dependence
- comonotonicity
- Dhaene, Denuit, Goovaerts, Kaas, Vyncke (2002a, 2002b, *IME*)
• Dhaene, Denuit, Goovaerts, Kaas, Vyncke (2002a, 2002b, *IME*)
• Kaas, Goovaerts, Dhaene and Denuit (2008, Modern Actuarial Risk Theory – Using R)
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Application 2: Losses Given Default

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Son born on June 25, 2014
Consider a portfolio of \( d \) obligors (loans, bonds and other financial instruments subject to default).

Classical model for the loss given default (LGD):

\[
L = \sum_{i=1}^{d} e_i \delta_i \mathbf{1}(X_i > a_i),
\]

- \( e_i \): positive deterministic exposure
- \( \delta_i \in (0, 1) \): percentage loss
- \( X_i \): loss variable of obligor \( i \)
- \( a_i \): default threshold
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- $\delta_i \in (0, 1)$: percentage loss
- $X_i$: loss variable of obligor $i$
- $a_i$: default threshold

This threshold model descends from Merton’s seminal firm-value work (Merton (1974, JF)).

The constant percentage loss ignores the severity of default.
A New Model

To take into account the severity of default, we introduce,

- **severity of default**: the percentage by which the loss variable exceeds the default threshold, that is,
  \[
  \left( \frac{X_i}{a_i} - 1 \right) +
  \]

- **loss function**: for each \( i \), define a loss function \( G_i \) that is non-decreasing with \( G_i(y) = 0 \) for \( y \leq 0 \) and \( G(\infty) = 1 \).
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The loss given default:

\[ L = \sum_{i=1}^{d} e_i G_i \left( \frac{X_i}{a_i} - 1 \right), \]

where the exposures are scaled such that \( \sum_{i=1}^{d} e_i = 1 \).
Similar ideas of introducing a loss function in the literature:

$$G(y) = (1 - e^{-\mu - \sigma y})^+.$$
Similar ideas of introducing a loss function in the literature:

- Pykhtin (2003, *RISK*) and Tasche (2004, *Working Paper*) employed such a loss function and even suggested the following specific form:

\[ G(y) = \left(1 - e^{-\mu - \sigma y}\right)_+.\]
Similar ideas of introducing a loss function in the literature:

- Pykhtin (2003, *RISK*) and Tasche (2004, *Working Paper*) employed such a loss function and even suggested the following specific form:

  \[ G(y) = (1 - e^{-\mu - \sigma y})_+ . \]

- Schuermann (2004, *Credit Risk: Models and Management*) illustrated that the distribution of the recovery rate, and hence the loss function, contains two modes.
The Catastrophic Edge

Our main focus is on the behavior of $L$ approaching its upper endpoint 1.

Understand the upper endpoint 1 – the catastrophic edge.
Our main focus is on the behavior of $L$ approaching its upper endpoint $1$.

Understand the upper endpoint $1$ – the catastrophic edge.

For this purpose, define

$$K = \frac{1}{1 - L},$$

which is a continuous and strictly increasing function of $L$ and, hence, can be thought of as a proxy for $L$. 
A Main Result

Theorem 4.1

Assume that the random vector \( \{X_1^+, \ldots, X_d^+\} \) follows an MRV structure with limit measure \( \nu \). Upon some regularity conditions it holds that

\[
\lim_{x \to \infty} \frac{P(K > x)}{F(G \leftarrow (1 - 1/x))} = \tilde{\nu}(A),
\]

where

\[
A = \left\{ y \in [0, \infty)^d : \sum_{i=1}^{d} e_i / y_i < 1 \right\}
\]

and \( \tilde{\nu} \) is another limit measure mainly determined by \( \nu \).
Numerical Results

We conduct numerical studies to examine the accuracy:

- **Gumbel copula** with $1 < \beta < \infty$
- Losses $X_1, X_2$ and $X_3$ identically distributed by Pareto($\alpha, \theta$)
- Loss function: $G(s) = 1 - (s + 1)^{-\gamma}$ for $s, \gamma > 0$.
- $(e_1, e_2, e_3) = (0.2, 0.4, 0.4)$
- $a_1 = a_2 = a_3 = a = 5$
- $c_1 = c_2 = c_3 = d_1 = d_2 = d_3 = 1$
- $\beta = 5$
- $\alpha = 2, \theta = 2$
- $\gamma = 0.8$

Marginal default probabilities calculated to be 0.082
Figure 4: Comparison of the two estimates of \( P(L > 1 - 1/x) \) (\( N = 10^7 \))
Figure 5: Comparison of the two estimates of $P(L > 1 - 1/x)$ ($N = 10^8$)
Figure 6: Comparison of the two estimators of $\text{VaR}_q[K]$ ($N = 10^7$)
Figure 7: Comparison of the two estimators of $\text{CTE}_q[K]$ ($N = 10^7$)
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Daughter born on March 28, 2013
Insurance Risk and Financial Risk

Consider an insurance company who invests its wealth in a financial market consisting of riskless and risky assets.
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This company is exposed to the following two risks:

- **insurance risk**: the traditional liability risk, namely insurance claims, related to the insurance portfolio
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- **insurance risk**: the traditional liability risk, namely insurance claims, related to the insurance portfolio
- **financial risk**: the asset risk related to the investment portfolio including inflation of economy and stock market crashes
Insurance Risk and Financial Risk

Consider an insurance company who invests its wealth in a financial market consisting of riskless and risky assets.

This company is exposed to the following two risks:

- **insurance risk**: the traditional liability risk, namely insurance claims, related to the insurance portfolio
- **financial risk**: the asset risk related to the investment portfolio including inflation of economy and stock market crashes

Both insurance risk and financial risk may impair the solvency of the insurance company.

We ask: which one of them plays a dominating role?
A Discrete-time Risk Model

- Denote by $X_n$ the insurer’s net loss within period $n$:
  \[ X_n = \text{claims} - \text{premiums} \]

- Then $X_1, X_2, \ldots$ correspond to the insurance risk
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- Then $X_1, X_2, \ldots$ correspond to the insurance risk

- Denote by $Y_n$ the “stochastic discount factor” over period $n$:
  \[ Y_n = \frac{1}{1 + R_n} \]

  with $R_n$ the stochastic return rate

- Then $Y_1, Y_2, \ldots$ correspond to the financial risk
The wealth of the insurance company accumulated at time $n$ is

$$W_n = x \prod_{j=1}^{n} (1 + R_j) + \sum_{i=1}^{n} (\text{premiums} - \text{claims})_i \prod_{j=i+1}^{n} (1 + R_j)$$

$$= \left( \prod_{j=1}^{n} (1 + R_j) \right) \left( x - \sum_{i=1}^{n} X_i \prod_{j=1}^{i} Y_j \right).$$
A Sum–Product Structure as a Risk Management Tool

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Comments on the sum–product structure:

$$S_n = \sum_{i=1}^{n} X_i \prod_{j=1}^{i} Y_j.$$
A Criticism on the Current Study

Assume that \((X_1, Y_1), (X_2, Y_2), \ldots \) are i.i.d. copies of \((X, Y)\). The mainstream of this study focuses on the case where:

- \(X\) has a Pareto-type tail
- \(Y\) is dominated by \(X\): \(\overline{G} = o(\overline{F})\)

References:
- Grey (1994, AAP)
- Frolova, Kabanov and Pergamenshchikov (2002, FS)
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Such financial catastrophes intensify the need to investigate the opposite case where

- $Y$ has a Pareto-type tail
- $X$ is dominated by $Y$

For this case, however, the sum–product structure $S_n$ becomes much more intractable.
Recall that a distribution $H$ on $(-\infty, \infty)$ is of regular variation if

$$\lim_{x \to \infty} \frac{H(xy)}{H(x)} = y^{-\alpha}, \quad y > 0, \ 0 < \alpha < \infty.$$ 

Strongly Regular Variation

Recall that a distribution $H$ on $(-\infty, \infty)$ is of regular variation if

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This does not yet enable us to derive an explicit asymptotic formula for the tail probability of $S_n$.

We strengthen the concept by using convolution equivalence so that $H$ has a finite $\alpha$th moment and we call it strongly regular variation.
The First Result

Assumption 5.1

*Every convex combination of $F$ and $G$, namely

$$pF + (1 - p)G, \quad 0 < p < 1,$$

is of strongly regular variation with index $0 < \alpha < \infty$.\*
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Theorem 5.1 (Li and T. (2014, B))

Assume that $X$ and $Y$ are independent, $EY^\alpha < 1$ and $E \ln (X_\land \lor 1) < \infty$. Then

$$P (S_\infty > x) \sim A_\infty \bar{F}(x) + B_\infty \bar{G}(x),$$

$$A_\infty = \frac{EY^\alpha}{1-EY^\alpha} \quad \text{and} \quad B_\infty = \frac{E(S_\infty \lor 0)^\alpha}{EY^\alpha (1-EY^\alpha)}.$$
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The Second Result

Assumption 5.2

1. \((X, Y)\) follows MRV\(_{-\alpha}\) for \(\alpha > 0\)
2. \(\nu(1, \infty] > 0\)
3. \(0 < EY^{\alpha/2} < 1\)

Hence, both risk variables are asymptotically dependent and tail equivalent.
Assumption 5.2

- \((X, Y)\) follows \(\text{MRV}_{-\alpha}\) for \(\alpha > 0\)
- \(\nu(1, \infty] > 0\)
- \(0 < \mathbb{E}Y^{\alpha/2} < 1\)

Hence, both risk variables are asymptotically dependent and tail equivalent.

Theorem 5.2 (Yuan (2014, In Progress))

It holds for \(A = \{(x, y) \in [0, \infty]^2 : xy > 1\}\) that

\[
P(S_\infty > x) \sim \left(1 + \frac{\mathbb{E}Y^{\alpha/2}}{1 - \mathbb{E}Y^{\alpha/2}}\right) \nu(A) \bar{F}(\sqrt{x}).
\]
Concluding Remarks

1. Brief Introduction
   - Why Extreme Value Theory
   - The Max-Domain of Attraction
   - Peaks over Threshold

2. Application 1: The Haezendonck–Goovaerts Risk Measure

3. Extreme Dependence
   - A General Framework
   - Some Models
   - Multivariate Regular Variation

4. Application 2: Losses Given Default

5. Application 3: Interplay of Insurance and Financial Risks

6. Concluding Remarks
In view of more and more frequent Black Swan events (such as global warming/polar vortex, financial crisis, and man-made disasters), there is an urgent need to analyze extreme risks in insurance and financial industry.

I illustrated that EVT is versatile in quantitative risk management by presenting its applications to various topics in the interdisciplinary area of insurance and finance. These topics include tail-related risk measures, losses given default in credit risk management, and the interplay of insurance and financial risks.

**SOA Proposal Request**

Tail Risk Analysis and Correlation of Risks in Tail/Extreme Environments

Qihe Tang (University of Iowa)
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1953 North Sea Flood
2008 Iowa City Flood
Smog in Beijing
Smog in Shanghai

Heavy smog seeps into Shanghai Hongqiao International Airport on the evening of Dec. 5, 2013.