The Sum–Product Structure as a Mechanism for Risk Management

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1. The Sum–Product Stochastic Structure

2. Modeling Insurer’s Wealth Process

3. A Bivariate AR(1) Model under Regular Variation
1. The Sum–Product Stochastic Structure
   1.1. The sum–product structure
   1.2. Insurance and financial risks
   1.3. Random difference equations

2. Modeling Insurer’s Wealth Process

3. A Bivariate AR(1) Model under Regular Variation
We are interested in the following \textit{sum-product structure}:

\[ L_n = \sum_{i=1}^{n} X_i \prod_{j=1}^{i} Y_j, \quad n \in \mathbb{N} \]  \hspace{1cm} (1)

- \( \{X_i\} \): real-valued random variables
- \( \{Y_j\} \): positive random variables
We are interested in the following **sum-product structure**:

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- \( \{Y_j\} \): positive random variables

In this stochastic structure, \( \{X_i\} \) and \( \{Y_j\} \) interact in a transparent way, providing great convenience and flexibility to introduce dependence structures to the study.

We show how the sum–product structure (1) **emerges naturally** in risk management for insurance and finance.
Insurance and Financial Risks

When an insurance company invests its wealth in a financial market, it is exposed to two kinds of risks:

- **insurance risk**: the traditional liability risk, namely insurance claims, related to the insurance portfolio
- **financial risk**: the asset risk related to the investment portfolio including inflations of economy and stock market crashes

Both risks may impair the solvency of the insurance company.

We shall show examples from risk modeling, in which \( \{ X_i \} \) and \( \{ Y_j \} \) in the sum–product structure (1) represent the two kinds of risks.
Literature Review

Norberg (1999, SPA)
Paulsen (1993, SPA)
Nyrhinen (1999, 2001, SPA)
Kalashnikov and Norberg (2002, SPA)
Frolova, Kabanov and Pergamenshchikov (2002, FS)
T. and Tsitsiashvili (2003, SPA)
Pergamenshchikov and Zeitouny (2006, SPA)
Paulsen (2008, PS)
Asmussen and Albrecher (2010, Ruin probabilities)
Yang and Konstantinides (2014, SAJ)

...
If \((X, Y), (X_1, Y_1), (X_2, Y_2), \ldots\), are i.i.d., then

\[
L_n \overset{d}{=} \sum_{i=1}^{n} X_i \prod_{j=i}^{n} Y_j
\]

\[
= \left( \sum_{i=1}^{n-1} X_i \prod_{j=i}^{n-1} Y_j + X_n \right) Y_n
\]

\[
\overset{d}{=} (L_{n-1} + X_n) Y_n.
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\]

The (weak) limit \(L_\infty\), if it exists, satisfies

\[
L_\infty \stackrel{d}{=} (L_\infty + X) Y,
\]

where \(L_\infty\) and \((X, Y)\) on the right-hand side are independent.
Kesten (1973, *AM*)

Vervaat (1979, *Advances in AP*)

Goldie (1991, *AAP*)

Grey (1994, *AAP*)

Goldie and Grübel (1996, *Advances in AP*)

Collamore (2009, *AAP*)

Enriquez, Sabot and Zindy (2009, *PTRF*)

Li and T. (2014, *B*)

:
1. The Sum–Product Stochastic Structure

2. Modeling Insurer’s Wealth Process
   2.1. The insurer’s wealth process
   2.2. The ruin probability
   2.3. The default probability

3. A Bivariate AR(1) Model under Regular Variation
Consider a discrete-time risk model. Within period $n$:

- the total premium income is denoted by a non-negative random variable $A_n$
- the total claim amount plus other daily costs is denoted by another non-negative random variable $B_n$
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- the total claim amount plus other daily costs is denoted by another non-negative random variable $B_n$

In the world **without economic factors**, the insurer’s wealth process $U_n$ exhibits a random walk structure:

$$U_0 = x > 0, \quad U_n = U_{n-1} + (A_n - B_n).$$
An Investment Portfolio

Suppose that a financial market consists of a risk-free bond with price $S_{0,n}$ and $d$ risky stocks with prices $S_{1,n}, \ldots, S_{d,n}$ at time $n$. 
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Denote by $\pi_{0,n}$ the proportion invested in the bond and by $\pi_{k,n}$ the proportion invested in stock $k$. Thus, $\pi_n = (\pi_{0,n}, \pi_{1,n}, \ldots, \pi_{d,n})$ is a stochastic process satisfying

$$\pi_n^\top \mathbf{1} = \pi_{0,n} + \pi_{1,n} + \cdots + \pi_{d,n} = 1$$

and other possible constraints.
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Denote by $V_n$ the value process of this investment portfolio. It holds that

$$\frac{V_n - V_{n-1}}{V_{n-1}} = \sum_{k=0}^{d} \pi_{k,n} \frac{S_{k,n} - S_{k,n-1}}{S_{k,n-1}},$$

or, equivalently,

$$\frac{V_n}{V_{n-1}} = \sum_{k=0}^{d} \pi_{k,n} \frac{S_{k,n}}{S_{k,n-1}}.$$
The insurer’s wealth process \( \{U_n^{(\pi)}\} \), starting with \( U_0^{(\pi)} = x > 0 \), evolves according to

\[
U_n^{(\pi)} = U_{n-1}^{(\pi)} \frac{V_n}{V_{n-1}} + (A_n - B_n) = U_{n-1}^{(\pi)} Y_{n-1} - X_n,
\]

where

- \( X_n = B_n - A_n \): the net loss over period \( n \), insurance risk
- \( Y_n = V_{n-1}/V_n \): the overall stochastic discount factor over period \( n \), financial risk

The quantity \( L_n^{(\pi)} \) denotes the stochastic present value of the insurer’s aggregate net losses. It exhibits a sum-product structure.
The Insurer’s Wealth Process

The insurer’s wealth process \( \{U_n^{(\pi)}\} \), starting with \( U_0^{(\pi)} = x > 0 \), evolves according to

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- \( X_n = B_n - A_n \): the net loss over period \( n \), insurance risk
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Iterating (3) yields

\[
U_n^{(\pi)} = \left( x - \sum_{i=1}^{n} X_i \prod_{j=1}^{i} Y_j \right) \left( \prod_{j=1}^{n} Y_j^{-1} \right) = \left( x - L_n^{(\pi)} \right) \left( \prod_{j=1}^{n} Y_j^{-1} \right).
\]

The quantity \( L_n^{(\pi)} \) denotes the stochastic present value of the insurer’s aggregate net losses. It exhibits a sum-product structure.
The classical ruin probability by time $T \leq \infty$ is

$$
\psi_{\pi}(x; T) = P \left( \inf_{0 \leq n \leq T} U_n^{(\pi)} < 0 \mid x \right)
= P \left( \inf_{0 \leq n \leq T} \left( x - L_n^{(\pi)} \right) \left( \prod_{j=1}^{n} Y_j^{-1} \right) < 0 \mid x \right)
= P \left( \sup_{0 \leq n \leq T} L_n^{(\pi)} > x \right),
$$

which is the tail probability of the maximum of $L_n^{(\pi)}$. 
The threshold 0 above does not make sense in practice. Insurance is a regulated business and it regulators will not allow an insurance company to continue its business if its wealth stays at a too low level.
The Default Probability

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Letting $a$ be a default threshold, the default probability by time $T \leq \infty$ is defined to be

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\psi_\pi(x; a, T) = P \left( \inf_{0 \leq n \leq T} U_n^{(\pi)} < a \ \mid \ x \right).
$$
The Default Probability

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For a certain value of $a$, it becomes the absolute ruin probability; see:

- Embrechts and Schmidli (1994, *Advances in AP*)
- Cai (2007, *Advances in AP*)
- Konstantinides, Ng and T. (2010, *JAP*)
- ...
Instead of (3) we use its equivalent form

\[ U_n^{(\pi)} - a = \left( U_{n-1}^{(\pi)} - a \right) Y_n^{-1} - (X_n - a Y_n^{-1} + a). \]

Write \( \tilde{X}_n = X_n - a Y_n^{-1} + a \). Iterating this recursive equation yields

\[ U_n^{(\pi)} - a = \left( x - a - \sum_{i=1}^{n} \tilde{X}_i \prod_{j=1}^{i} Y_j \right) \prod_{j=1}^{n} Y_j^{-1} = \left( x - a - \tilde{L}_n^{(\pi)} \right) \prod_{j=1}^{n} Y_j^{-1}. \]

Note that \( \tilde{L}_n^{(\pi)} \) appearing above also exhibits a sum-product structure.
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Note that \( \tilde{L}_n^{(\pi)} \) appearing above also exhibits a sum-product structure.

Thus,

\[ \psi_{\pi}(x; a, T) = P \left( \sup_{0 \leq n \leq T} \tilde{L}_n^{(\pi)} > x - a \right). \]

The dependence structure between \( \tilde{X}_n \) and \( Y_n \) becomes complex.
1. The Sum–Product Stochastic Structure

2. Modeling Insurer’s Wealth Process

3. A Bivariate AR(1) Model under Regular Variation
   3.1. Regularly varying distributions
   3.2. A bivariate AR(1) risk model
   3.3. Our main result
   3.4. Portfolio optimization
For a distribution $F$ on $\mathbb{R}$, we write $F \in \mathcal{R}_{-\alpha}$ for some $\alpha \geq 0$ if

$$\bar{F}(xy) \sim y^{-\alpha} \bar{F}(x), \quad y > 0.$$ 

The union

$$\mathcal{R} = \bigcup_{\alpha \geq 0} \mathcal{R}_{-\alpha}$$

forms one of the most important classes of heavy-tailed distributions.

- Bingham, Goldie and Teugels (1987, *Regular Variation*)
- Resnick (1987, *Extreme Values, Regular Variation, and Point Processes*)
Denote the claim amount (plus other expenses) paid by an insurer within period $i$ by a nonnegative random variable $B_i$

Assume that these claim amounts form an AR(1) process: starting with a deterministic value $B_0 \geq 0$,

$$B_i = \rho B_{i-1} + \zeta_i,$$

(4)

- the autoregressive coefficient $\rho$ takes value in $[0, 1)$
- innovations $\{\zeta_i\}$ are i.i.d. copies of a nonnegative random variable $\zeta$
Claims Follow AR(1)

- Denote the claim amount (plus other expenses) paid by an insurer within period \( i \) by a nonnegative random variable \( B_i \)
- Assume that these claim amounts form an AR(1) process: starting with a deterministic value \( B_0 \geq 0 \),

\[
B_i = \rho B_{i-1} + \zeta_i, \tag{4}
\]

- the autoregressive coefficient \( \rho \) takes value in \([0, 1)\)
- innovations \( \{\zeta_i\} \) are i.i.d. copies of a nonnegative random variable \( \zeta \)

An advantage of the AR(1) model is that it can capture asymptotic dependence between claim amounts:

**Lemma**

*If \( F \in \mathcal{R}_{-\alpha} \) for some \( \alpha > 0 \), then it holds for all \( i_1, i_2 \in \mathbb{N} \) that*

\[
\lim_{x \to \infty} P(B_{i_2} > x | B_{i_1} > x) = \rho^{i_2 - i_1} \alpha.
\]
Suppose that there is a discrete-time financial market consisting of:

- a risk-free bond with a deterministic continuously compounded rate of interest $r > 0$
- a risky stock with a stochastic log-return rate $R_i \in \mathbb{R}$ during period $i$
Suppose that there is a discrete-time financial market consisting of:
- a risk-free bond with a deterministic continuously compounded rate of interest $r > 0$
- a risky stock with a stochastic log-return rate $R_i \in \mathbb{R}$ during period $i$

These log-return rates are also assumed to follow an AR(1) process:
starting with a deterministic value $R_0$,

$$ (R_i - \mu_R) = \gamma (R_{i-1} - \mu_R) + \eta_i, $$

the autoregressive coefficient $\gamma$ takes value in $(-1, 1)$
the innovations $\{\eta_i\}$ are i.i.d. copies of a real-valued random variable $\eta$ with mean 0
the constant $\mu_R$ is the mean of the stationary solution $R_\infty$
Assume that \( \{ \xi_i \} \) and \( \{ \eta_i \} \) are mutually independent and so are the two AR(1) processes (4) and (5).

Suppose that at the beginning of each period the insurer invests a fixed proportion \( \pi \in [0, 1] \) in the stock and keeps the rest in the bond. Then the wealth process \( \{ U_m^{(\pi)} \} \) evolves according to

\[
U_m^{(\pi)} = \left( (1 - \pi)e^r + \pi e^{R_m} \right) U_{m-1}^{(\pi)} - (B_m - a),
\]

- \( U_0^{(\pi)} = x > 0 \): a deterministic initial value
- \( a > 0 \): a constant premium amount during each period
Cai (2002, *JAP*)


Gerber (1982, *IME*)

Mikosch and Samorodnitsky (2000, *AAP*)

Wilkie (1986, *Transactions of the Faculty of Actuaries*)

Wilkie (1987, *IME*)

Yang and Zhang (2003, *PEIS*)
Our Main Result

Theorem (T. and Yuan (2012, NAAJ))

Assume that $F \in \mathcal{R}_{-\alpha}$ for some $\alpha > 0$. Then it holds for every $n \in \mathbb{N}$ that

$$
\psi(x; n) \sim E \left[ F \left( x + \sum_{i=1}^{n} \theta_i \left( a - \rho^i B_0 \right) \right) \sum_{j=1}^{n} \left( \sum_{i=j}^{n} \theta_i \rho^{i-j} \right)^{\alpha} \right].
$$
- $\xi$ in (4) follows Pareto with shape parameter $\alpha > 1$
- $\eta$ in (5) follows normal with mean 0 and variance $\sigma^2$
- The parameters are set to
  - $\pi = 0.2, 0.5$ or $0.8$
  - $n = 4$
  - $\alpha = 1.1, 1.3$ or $1.5$
  - $\rho = 0.3$ or $0.5$
  - $B_0 = 5.0$, $E[B] = 5.0$, $E[\xi] = (1 - \rho)E[B]$
  - $a = 5.5$
  - $r = 1.242\%$ (so that $e^r - 1 = 1.25\%$)
  - $\gamma = 0.5$ or $0.8$
  - $R_0 = 1.5\%$, $E[R] = 1.5\%$, $\sigma = 0.2$
- The sample size is $N = 1,000,000$ for both simulations and asymptotics despite the fact that it is more than enough for asymptotics.
Graph 4.1(a)  Accuracy of the asymptotic estimate $\psi_2$ for $\pi = 0.2$

The graph shows the accuracy of the asymptotic estimate $\psi_2$ for different values of $\alpha$, $\rho$, and $\gamma$. The curves represent different combinations of these parameters:

- $\alpha = 1.1, \rho = 0.3, \gamma = 0.8$
- $\alpha = 1.3, \rho = 0.5, \gamma = 0.5$
- $\alpha = 1.5, \rho = 0.5, \gamma = 0.5$

The x-axis represents initial wealth, and the y-axis represents the ruin probability. The right graph shows the ratio $\psi_2 / \psi_1$ for the same parameter combinations.
Graph 4.1(b) Accuracy of the asymptotic estimate $\psi_2$ for $\pi = 0.5$

$\psi_1$, $\psi_2$

$\alpha = 1.1$, $\rho = 0.3$, $\gamma = 0.8$
$\alpha = 1.3$, $\rho = 0.5$, $\gamma = 0.5$
$\alpha = 1.5$, $\rho = 0.5$, $\gamma = 0.5$

Initial wealth

Ruin probability

$\psi_2 / \psi_1$
Graph 4.1(c) Accuracy of the asymptotic estimate $\psi_2$ for $\pi = 0.8$

$\psi_1$  
$\psi_2$

$\alpha = 1.1, \rho = 0.3, \gamma = 0.8$
$\alpha = 1.3, \rho = 0.5, \gamma = 0.5$
$\alpha = 1.5, \rho = 0.5, \gamma = 0.5$

Initial wealth

Ruin probability

$\psi_2 / \psi_1$

Initial wealth
Graph 4.2(a)  Accuracy of the asymptotic estimate $\psi_2$ for $\alpha = 2.0$ (N = 1,000,000)
Graph 4.2(b)  Accuracy of the asymptotic estimate $\psi_2$ for $\alpha = 2.0$ (N = 10,000,000)
Generally speaking, investors aim at maximizing gains while minimizing risks. How to balance between gains and risks is a question that permeates many areas of finance.
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In the insurance context, due to the increasing prudence of insurance regulations, a solvency constraint needs to be imposed in portfolio optimization problems.
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In the insurance context, due to the increasing prudence of insurance regulations, a solvency constraint needs to be imposed in portfolio optimization problems.

Our goal is to determine a value of $\pi$ that maximizes the expected terminal wealth subject to a constraint on $\psi(x; n)$:

$$\begin{align*}
\arg \max_{0 \leq \pi \leq 1} E[U_n^{(\pi)}], \\
\text{subject to } \psi(x; n) \leq 1 - q,
\end{align*}$$

where $0 < q < 1$ is chosen to be close to 1, say, $q = 0.995$. 
Literature Review

- Paulsen (2003, FS)
- Irgens and Paulsen (2005, SAJ)
- Dickson and Drekic (2006, AAS)
- Kostadinova (2007, IME)
- He, Hou, and Liang (2008, IME)
\( \xi \) in (4) follows Pareto with shape parameter \( \alpha > 1 \)

\( \eta \) in (5) follows normal with mean 0 and variance \( \sigma^2 \)

The parameters are set to

- \( n = 4 \)
- \( \alpha = 1.1, 1.5 \) or \( 2.0 \)
- \( \rho = 0.3 \)
- \( B_0 = 5.0, \ E[B] = 5.0, \ E[\xi] = (1 - \rho)E[B] \)
- \( a = 5.5 \)
- \( r = 1.242\% \)
- \( \gamma = 0.8, \)
- \( R_0 = 1.5\%, \ E[R] = 1.5\% \) or \( 3\% \)
- \( \sigma = 0.2. \)
Graph 5.1 The optimal $\pi$ for different values of the initial wealth

- $\alpha = 1.1$, $\mu_R = 1.5\%$, $\mu_R = 3.0\%$
- $\alpha = 1.5$, $\mu_R = 1.5\%$, $\mu_R = 3.0\%$
- $\alpha = 2.0$, $\mu_R = 1.5\%$, $\mu_R = 3.0\%$
Thank You Very Much!!!