Reducing Risk in Convex Order

Qihe Tang (University of Iowa)

Based on a joint work with
Junnan He (Washington University in St. Louis) and
Huan Zhang (University of Iowa)

The 50th Actuarial Research Conference (ARC), University of Toronto, Toronto, Canada, August 5–8, 2015
Outline

1. Overall Descriptions
   - Goal of Our Study
   - Risk Reducer

2. Characterization of General Risk Reducers
   - Why General Risk Reducers
   - Main Result

3. Fully Dependent Risk Reducers
   - Main Result
   - Application to Index-linked Hedging Strategies

4. Universal Risk Reducers
   - In the Unidimensional Case
   - Extension to the Multidimensional Case

5. Concluding Remarks
Outline

1. Overall Descriptions
   - Goal of Our Study
   - Risk Reducer

2. Characterization of General Risk Reducers
   - Why General Risk Reducers
   - Main Result

3. Fully Dependent Risk Reducers
   - Main Result
   - Application to Index-linked Hedging Strategies

4. Universal Risk Reducers
   - In the Unidimensional Case
   - Extension to the Multidimensional Case

5. Concluding Remarks
Goal of Our Study

Consider a portfolio with random loss $X$ at the end of a given reference period. In order to have a better control of the risk involved, it is often desirable that an additional asset with random loss $Z$ is added to the position $X$ so that the overall risk is reduced.

That is,

$$X + Z \text{ is less risky than } X + E[Z].$$
Consider a portfolio with random loss $X$ at the end of a given reference period. In order to have a better control of the risk involved, it is often desirable that an additional asset with random loss $Z$ is added to the position $X$ so that the overall risk is reduced.

That is,

$$X + Z \text{ is less risky than } X + E[Z].$$

Here we mean to reduce the risk in convex order.

Note that many popular risk measures are consistent with convex order. Thus, we conduct our study in terms of convex order rather than risk measures.
Convex Order

For two risks $Y_1$ and $Y_2$, say $Y_1$ is less risky than $Y_2$ in convex order, denoted by $Y_1 \leq_{cx} Y_2$, if

$$E[v(Y_1)] \leq E[v(Y_2)]$$

for every convex function $v$ such that the two expectations exist.

References:
- Denuit-Dhaene-Goovaerts-Kaas (2005, Actuarial Theory for Dependent Risks)
- Shaked-Shanthikumar (2007, Stochastic Orders)
For two risks $Y_1$ and $Y_2$, say $Y_1$ is less risky than $Y_2$ in convex order, denoted by $Y_1 \leq_{cx} Y_2$, if

$$E [\nu(Y_1)] \leq E [\nu(Y_2)]$$

for every convex function $\nu$ such that the two expectations exist.

References:

- Shaked-Shanthikumar (2007, *Stochastic Orders*)

Closely related terminologies include: second-order stochastic dominance, the Rothschild–Stiglitz increase in risk, majorization, mean preserving spread, stop-loss order, ...
Risk Reducer

Definition (Cheung-Dhaene-Lo-Tang (2014, IME))
For a given random variable $X$, a random variable $Z$ is said to be its risk reducer, denoted by $Z \in R(X)$, if

$$X + Z \leq_{cx} X + E[Z].$$

(1)
Definition (Cheung-Dhaene-Lo-Tang (2014, IME))

For a given random variable $X$, a random variable $Z$ is said to be its risk reducer, denoted by $Z \in R(X)$, if

$$X + Z \leq_{cx} X + E[Z].$$  \hfill (1)

We aim at a structural characterization of the set $R(X)$. 
On Strassen’s Theorem

The classical Strassen’s martingale characterization states that $Y_1 \leq_{cx} Y_2$ if and only if there exist random variables $X_1 =_d Y_1$ and $X_2 =_d Y_2$ such that $X_1 =_{a.s.} E[X_2|X_1]$.

Reference:

- Strassen (1965, AMS)
- Bäuerle-Müller (2006, IME)
- Denuit-Dhaene-Goovaerts-Kaas (2005, Actuarial Theory for Dependent Risks)

Using this we can easily give a characterization for $R(X)$.
On Strassen’s Theorem

The classical Strassen’s martingale characterization states that $Y_1 \leq_{cx} Y_2$ if and only if there exist random variables $X_1 =_{d} Y_1$ and $X_2 =_{d} Y_2$ such that $X_1 =_{a.s.} E[X_2|X_1]$.

Reference:

- Strassen (1965, AMS)
- Bäuerle-Müller (2006, IME)

Using this we can easily give a characterization for $R(X)$.

However, the characterization that we pursue will be different from and more structural than Strassen’s characterization.
Outline

1. Overall Descriptions
   - Goal of Our Study
   - Risk Reducer

2. Characterization of General Risk Reducers
   - Why General Risk Reducers
   - Main Result

3. Fully Dependent Risk Reducers
   - Main Result
   - Application to Index-linked Hedging Strategies

4. Universal Risk Reducers
   - In the Unidimensional Case
   - Extension to the Multidimensional Case

5. Concluding Remarks
Two random variables $Y_1$ and $Y_2$ are said to be (a.s.) comonotonic if there is a null set $N$ such that

\[(Y_1(\omega) - Y_1(\omega')) (Y_2(\omega) - Y_2(\omega')) \geq 0, \quad \forall \omega, \omega' \in \Omega \setminus N.\]

The two random variables $Y_1$ and $Y_2$ are said to be (a.s.) counter-monotonic if $Y_1$ and $-Y_2$ are comonotonic.
Counter-monotonic Risk Reducers

Denote by $H(X)$ the collection of all random variables $Z$ which are counter-monotonic with the combined position $X + Z$:

$$H(X) = \{ Z : (Z, X + Z) \text{ is counter-monotonic} \}. \quad (2)$$

**Theorem (Cheung-Dhaene-Lo-Tang (2014, IME))**

*Any element in $H(X)$ is a counter-monotonic risk reducer for $X$.***
Counter-monotonic Risk Reducers

Denote by $H(X)$ the collection of all random variables $Z$ which are counter-monotonic with the combined position $X + Z$:

$$H(X) = \{ Z : (Z, X + Z) \text{ is counter-monotonic} \}. \quad (2)$$

**Theorem (Cheung-Dhaene-Lo-Tang (2014, IME))**

Any element in $H(X)$ is a counter-monotonic risk reducer for $X$.

A function $f$ is said to be 1-Lipschitz on $D$ if

$$|f(x) - f(y)| \leq |x - y|, \quad \forall x, y \in D.$$ 

**Corollary (Cheung-Dhaene-Lo-Tang (2014, IME))**

$Z \in H(X)$ if and only if there exists a function $h$ which is non-decreasing and 1-Lipschitz on the support of $X$ such that $Z = a.s. - h(X)$. 
Recall the definition of risk reducer. Apparently, in order for inequality (1) to hold, it is not necessary to require $Z$ to be counter-monotonic with $X$. 

Example
Restricted to the space of normal distributions, it is easy to see that (1) holds if $X$ and $Z$ jointly follow a bivariate normal distribution with correlation coefficient $\rho$ satisfying $-1 \leq \rho \leq -\frac{1}{\sqrt{\text{Var}[Z] \text{Var}[X]}}$, yielding a much broader range than counter-monotonicity.
Recall the definition of risk reducer. Apparently, in order for inequality (1) to hold, it is not necessary to require $Z$ to be counter-monotonic with $X$.

**Example**

Restricted to the space of normal distributions, it is easy to see that (1) holds if $X$ and $Z$ jointly follow a bivariate normal distribution with correlation coefficient $\rho$ satisfying

$$-1 \leq \rho \leq -\frac{1}{2} \sqrt{\frac{\text{Var}[Z]}{\text{Var}[X]}},$$

yielding a much broader range than counter-monotonicity.
Why General Risk Reducers

Example

Consider a bundle of home and auto insurance in which the potential monetary losses on the home and auto are $X_1$ and $X_2$, respectively. To decide the feasibility of the corresponding indemnity payoffs $I_1(X_1)$ and $I_2(X_2)$, we hope that $Z = -(I_1(X_1) + I_2(X_2))$ is a risk reducer for $X = X_1 + X_2$. Note that $Z$ is not counter-monotonic with $X$ in general.
The set of all random variables that are identically distributed as $X$:

$$D(X) = \{ X' : X' =_d X \}$$

The convex hull of a set $A$:

$$\text{Conv}(A) = \left\{ \sum_{i=1}^{m} a_i X_i : m \in \mathbb{N}, \ a_i \geq 0, \ \sum_{i=1}^{m} a_i = 1 \text{ and } X_i \in A \right\}$$

Denote by $\overline{\text{Conv}}(A)$ the closure of $\text{Conv}(A)$ in $L^1$ space.
A probability measure $P$ is **atomless** if for any measurable set $A$ with $P(A) > 0$ there exists a measurable subset $B$ such that $0 < P(B) < P(A)$. 
Main Result

A probability measure $P$ is atomless if for any measurable set $A$ with $P(A) > 0$ there exists a measurable subset $B$ such that $0 < P(B) < P(A)$.

**Theorem**

Let $(\Omega, \mathcal{B}_\Omega, P)$ be an atomless and standard Borel space. Then

$$R(X) = \overline{\text{Conv}(D(X))} - X + \mathbb{R}.$$
Outline

1. Overall Descriptions
   - Goal of Our Study
   - Risk Reducer

2. Characterization of General Risk Reducers
   - Why General Risk Reducers
   - Main Result

3. Fully Dependent Risk Reducers
   - Main Result
   - Application to Index-linked Hedging Strategies

4. Universal Risk Reducers
   - In the Unidimensional Case
   - Extension to the Multidimensional Case

5. Concluding Remarks
Deterministic transformation of risks has been studied by, among others:

- Meyer and Ormiston (1989, *JRU*)
- Quiggin (1991, *JRU*)

A fully dependent risk reducer need not be a counter-monotonic one. Example

Let $X$ be uniformly distributed on $[0, 1]$ and let $Z = \alpha \sin(2\pi X)$.

Obviously, $Z$ is not counter-monotonic with $X$. One can check that $Z$ is a risk reducer for $X$ for $\alpha > 0$ small enough.
Deterministic transformation of risks has been studied by, among others:

- Meyer and Ormiston (1989, *JRU*)
- Quiggin (1991, *JRU*)

A fully dependent risk reducer need not be a counter-monotonic one.

**Example**

Let $X$ be uniformly distributed on $[0, 1]$ and let $Z = \alpha \sin(2\pi X)$. Obviously, $Z$ is not counter-monotonic with $X$. One can check that $Z$ is a risk reducer for $X$ for $\alpha > 0$ small enough.
Main Result

Consider the set of all risk reducers of the form $Z =_{\text{a.s.}} h(X)$ for some measurable but not necessarily monotone function $h$:

$$\tilde{R}(X) = \{ Z =_{\text{a.s.}} h(X) : X + Z \leq_{\text{cx}} X + E[Z] \}.$$
Main Result

Consider the set of all risk reducers of the form $Z =_{\text{a.s.}} h(X)$ for some measurable but not necessarily monotone function $h$:

$$\tilde{R}(X) = \{ Z =_{\text{a.s.}} h(X) : X + Z \leq_{\text{cx}} X + E[Z] \}.$$ 

Similarly as before, define

$$\tilde{D}(X) = \{ X' \text{ defined on } (\Omega, \sigma(X), P) : X' =_{\text{d}} X \}.$$
Main Result

Consider the set of all risk reducers of the form $Z =_{\text{a.s.}} h(X)$ for some measurable but not necessarily monotone function $h$:

$$\tilde{R}(X) = \{Z =_{\text{a.s.}} h(X) : X + Z \leq_{cx} X + E[Z]\}.$$ 

Similarly as before, define

$$\tilde{D}(X) = \{X' \text{ defined on } (\Omega, \sigma(X), P) : X' =_{\text{d}} X\}.$$ 

Theorem

Let $(\Omega, B_\Omega, P)$ be an atomless and standard Borel space, and let the risk $X$ follow a continuous distribution $F$. Then

$$\tilde{R}(X) = \overline{\text{Conv}}(\tilde{D}(X)) - X + \mathbb{R}.$$
Index-linked Hedging Strategies

Index-linked hedging strategies link the payoff of a contract to the development of an index.

Despite its benefits such as high transparency, low transaction costs, and reduction of moral hazard, the use of an index leads to basis risk, as the insurer’s exposure is usually not fully dependent on the index.

It is important to control the basis risk to an admissible level.
Index-linked Hedging Strategies

Index-linked hedging strategies link the payoff of a contract to the development of an index.

Despite its benefits such as high transparency, low transaction costs, and reduction of moral hazard, the use of an index leads to basis risk, as the insurer’s exposure is usually not fully dependent on the index.

It is important to control the basis risk to an admissible level.

References:

- Gatzert-Kellner (2013, *JRI*)
Application to Index-linked Hedging Strategies

Consider $n$ insurers in a certain state having unhedged losses $L_1, \ldots, L_n$. The sum $L = \sum_{j=1}^{n} L_j$ represents a **statewide industry loss index**.

A hedging strategy for insurer $i$ under the loss index $L$ defines a function $h(L)$. Then the hedged loss becomes $L^S_i = L_i - h(L)$. It is natural to require that the hedged loss $L^S_i$ is less risky, or, more precisely, $Z = -h(L)$ is a risk reducer for $L_i$. 

**Corollary** Suppose that $(L_1, \ldots, L_n)$ follows a multivariate normal distribution with positive definite covariance matrix $[s_{ij}]_{n \times n}$. Then all such hedgers $h(L)$ form the set $-\sum_{j=1}^{n} s_{ij} \sqrt{\sum_{i,j=1}^{n} s_{ij}} (\text{Conv} (\tilde{D} (\varepsilon))) - \varepsilon) + R$, where $\varepsilon = (L - E[L]) / \text{Var}(L)$. 

Qihe Tang (University of Iowa) 
Reducing Risk in Convex Order 
ARC 2015
Consider \( n \) insurers in a certain state having unhedged losses \( L_1, \ldots, L_n \). The sum \( L = \sum_{j=1}^{n} L_j \) represents a statewide industry loss index.

A hedging strategy for insurer \( i \) under the loss index \( L \) defines a function \( h(L) \). Then the hedged loss becomes \( L_i^S = L_i - h(L) \). It is natural to require that the hedged loss \( L_i^S \) is less risky, or, more precisely, \( Z = -h(L) \) is a risk reducer for \( L_i \).

**Corollary**

Suppose that \((L_1, \ldots, L_n)\) follows a multivariate normal distribution with positive definite covariance matrix \([s_{ij}]_{n \times n}\). Then all such hedgers \( h(L) \) form the set

\[
- \frac{\sum_{j=1}^{n} s_{ij}}{\sqrt{\sum_{i,j=1}^{n} s_{ij}}} \left( \text{Conv}(\tilde{D}(\varepsilon)) - \varepsilon \right) + \mathbb{R},
\]

where \( \varepsilon = (L - E[L]) / \text{Var}(L) \).
Outline

1. Overall Descriptions
   - Goal of Our Study
   - Risk Reducer

2. Characterization of General Risk Reducers
   - Why General Risk Reducers
   - Main Result

3. Fully Dependent Risk Reducers
   - Main Result
   - Application to Index-linked Hedging Strategies

4. Universal Risk Reducers
   - In the Unidimensional Case
   - Extension to the Multidimensional Case

5. Concluding Remarks
In the Unidimensional Case

Usually, the decision maker knows about the payoff function $X$, but not the underlying probability measure.

We pursue conditions under which $Z$ is always a risk reducer for a given random variable $X$ under all underlying probability measures. Such a random variable $Z$ is called a universal risk reducer for $X$. 

Recall the set $H(X)$ defined in (2).

Theorem
Let $(\Omega, F)$ be a measurable space in which every singleton $\{\omega\} \subset \Omega$ is measurable and let $X$ be a given random variable. Then $Z \in H(X)$ under every probability measure $P$ such that $E_P[|X|] < \infty$ if and only if $Z \in H(X)$. 

Qihe Tang (University of Iowa)  
Reducing Risk in Convex Order  
ARC 2015 21 / 24
In the Unidimensional Case

Usually, the decision maker knows about the payoff function $X$, but not the underlying probability measure.

We pursue conditions under which $Z$ is always a risk reducer for a given random variable $X$ under all underlying probability measures. Such a random variable $Z$ is called a universal risk reducer for $X$.

Recall the set $H(X)$ defined in (2).

**Theorem**

Let $(\Omega, \mathcal{F})$ be a measurable space in which every singleton $\{\omega\} \subset \Omega$ is measurable and let $X$ be a given random variable.
In the Unidimensional Case

Usually, the decision maker knows about the payoff function $X$, but not the underlying probability measure.

We pursue conditions under which $Z$ is always a risk reducer for a given random variable $X$ under all underlying probability measures. Such a random variable $Z$ is called a universal risk reducer for $X$.

Recall the set $H(X)$ defined in (2).

**Theorem**

Let $(\Omega, \mathcal{F})$ be a measurable space in which every singleton $\{\omega\} \subset \Omega$ is measurable and let $X$ be a given random variable. Then $Z \in R(X)$ under every probability measure $P$ such that $E_P[|X|] < \infty$ if and only if $Z \in H(X)$. 
We are now interested in a situation where
- the given random position is decided by multiple risk factors, but
- risk reducers can only be constructed based on one of the risk factors (e.g., due to the limitation of knowledge or a certain regulatory requirement).

We ask under what conditions such a risk reducer reduces the overall risk.
Extension to the Multidimensional Case

We are now interested in a situation where

- the given random position is decided by **multiple risk factors**, but
- risk reducers can only be constructed based on one of the risk factors (e.g., due to the limitation of knowledge or a certain regulatory requirement).

We ask under what conditions such a risk reducer reduces the overall risk.

Restrict to two dimensions:

**Theorem**

For \( i = 1, 2 \), let \((\Omega_i, \mathcal{F}_i)\) be a measurable space in which every singleton \( \{\omega_i\} \subset \Omega_i \) is measurable. Let \( X : \Omega_1 \times \Omega_2 \rightarrow \mathbb{R} \) and \( Z : \Omega_2 \rightarrow \mathbb{R} \) be two random variables.

Then \( Z \) is a risk reducer for \( X \) under every product probability measure \( P = P_1 \times P_2 \) such that

\[
E_P[|X|] < \infty \quad \text{and} \quad E_P[|Z|] < \infty
\]

if and only if

\[ Z \in \bigcap_{\omega_1 \in \Omega_1} H (X_{\omega_1}) . \]
We are now interested in a situation where

- the given random position is decided by multiple risk factors, but
- risk reducers can only be constructed based on one of the risk factors (e.g., due to the limitation of knowledge or a certain regulatory requirement).

We ask under what conditions such a risk reducer reduces the overall risk.

Restrict to two dimensions:

**Theorem**

For $i = 1, 2$, let $(\Omega_i, \mathcal{F}_i)$ be a measurable space in which every singleton $\{\omega_i\} \subset \Omega_i$ is measurable. Let $X : \Omega_1 \times \Omega_2 \to \mathbb{R}$ and $Z : \Omega_2 \to \mathbb{R}$ be two random variables. Then $Z$ is a risk reducer for $X$ under every product probability measure $P = P_1 \times P_2$ such that $E_P[|X|] < \infty$ and $E_P[|Z|] < \infty$ if and only if $Z \in \bigcap_{\omega_1 \in \Omega_1} H(X^{\omega_1})$. 
1. **Overall Descriptions**
   - Goal of Our Study
   - Risk Reducer

2. **Characterization of General Risk Reducers**
   - Why General Risk Reducers
   - Main Result

3. **Fully Dependent Risk Reducers**
   - Main Result
   - Application to Index-linked Hedging Strategies

4. **Universal Risk Reducers**
   - In the Unidimensional Case
   - Extension to the Multidimensional Case

5. **Concluding Remarks**
Concluding Remarks

- Reviewed the concept of risk reducer in convex order.
- By using the concept of convex hull, gave a constructive characterization of risk reducers in an atomless probability space.
- Characterized fully dependent risk reducers.
- In both unidimensional and multidimensional cases, characterized universal risk reducers regardless of the underlying probability measure.
- An application to index-linked hedging strategies was proposed.
Concluding Remarks

- Reviewed the concept of risk reducer in convex order.
- By using the concept of convex hull, gave a constructive characterization of risk reducers in an atomless probability space.
- Characterized fully dependent risk reducers.
- In both unidimensional and multidimensional cases, characterized universal risk reducers regardless of the underlying probability measure.
- An application to index-linked hedging strategies was proposed.

Thank You for Listening!