Losses Given Default in the Presence of Extreme Risks

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1. Modeling Losses Given Default
   - A Classical Model
   - A New Model

2. Multivariate Regular Variation
   - Motivating Discussions
   - Definition
   - On the Limit Measure

3. The Main Result
   - Our Goal
   - A Heuristic Consideration
   - Approximating the Distribution of LGD
   - Numerical Studies

4. Concluding Remarks
Outline

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4. Concluding Remarks
A Classical Model for LGD

Consider a portfolio of $d$ obligors subject to default (loans, bonds and other financial instruments).

Classical model for the LGD:

$$L = \sum_{i=1}^{d} e_i \delta_i \mathbf{1}(X_i > F_i(1-p))$$ (1)

- $e_i$: positive deterministic exposure
- $\delta_i \in (0, 1)$: percentage loss
- $X_i$: loss variable of obligor $i$
- $p \in (0, 1)$: exogenously given default probability
- $F_1(1-p), \ldots, F_d(1-p)$: individual default thresholds
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This threshold model descends from Merton’s seminal firm-value work (Merton (1974, JF)).
The constant percentage loss *ignores* the severity of default.
To remedy this, we introduce

- severity of default: the percentage by which the loss variable exceeds the default threshold, that is,

\[ S_i = \left( \frac{X_i}{a_i} - 1 \right)_+ \]

- loss settlement function: a loss function \( G_i \) that is non-decreasing with \( G_i(s) = 0 \) for \( s \leq 0 \) and \( G(\infty) = 1 \)
Similar ideas of introducing a loss settlement function:

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- Pykhtin (2003, *RISK*) and Tasche (2004, *Working Paper*) employed such a loss settlement function and even suggested the following specific form:

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  \[ G(y) = (1 - e^{-\mu - \sigma y})_+ . \]

- Schuermann (2004, *Credit Risk: Models and Management*) illustrated that the distribution of the recovery rate, and hence the loss function, contains two modes.
Our new model for the LGD:

\[ L(p) = \sum_{i=1}^{d} e_i G_i \left( \frac{X_i}{F_i^{-}(1-p)} - 1 \right), \quad (2) \]

where the exposures are scaled such that

\[ \sum_{i=1}^{d} e_i = 1. \]

Compare this new model (2) with the classical model (1).
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Motivating Discussions

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It turns out that both challenges may be dealt with in a unified framework called multivariate regular variation.
Definition

A random vector $\mathbf{X} = (X_1, \ldots, X_d)$ is said to have an MRV structure if there exists a non-degenerate limit measure $\nu$ such that, for some distribution function $F$ and every $\nu$-continuous set $A \subset [0, \infty)^d$ away from 0,

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\lim_{x \to \infty} \frac{1}{F(x)} \Pr (\mathbf{X} \in xA) = \nu (A).
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Multivariate Regular Variation

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References:

- de Haan and Ferreira (2006, *Extreme Value Theory*)
- Resnick (2007, *Heavy-Tail Phenomena*)
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T. and Yuan (2013, *NAAJ*) showed various commonly-used examples including:

- linear combinations
- mixtures
- Archimedean copulas
The definition implies that the limit measure $\nu$ is **homogeneous**: for some index $\alpha \geq 0$,

$$\nu(tB) = t^{-\alpha} \nu(B) \quad \text{for all } B \in \mathcal{B}.$$ 

Hence, we write $X \in \text{MRV}_{-\alpha}$.
The limit measure $\nu$ carries all asymptotic dependence information of $X$ in the upper-right tail:

$$\lim_{x \to \infty} \frac{1}{F(x)} \Pr \left( \bigcap_{i=1}^{d} (X_i > x) \right) = \nu(1, \infty).$$

Asymptotic dependence: $\nu(1, \infty) > 0$ means that $X$ exhibits large joint movements. If $\nu$ is concentrated on a straight line, then the components of $X$ are asymptotically fully dependent, of which comonotonicity is a special case.

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On the Limit Measure - Tail Dependence

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4. Concluding Remarks
Recall the LGD given in (2):

\[ L(p) = \sum_{i=1}^{d} e_i G_i \left( \frac{X_i}{F_i^{-1}(1-p)} - 1 \right). \]

We study \( \Pr(L(p) > l) \) as \( p \downarrow 0 \) for arbitrarily fixed \( l \in (0, 1) \).
Our Goal

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- The default probability \( p \) being small means that the portfolio consists of assets of good credit quality.
- The extreme scenario \( p \downarrow 0 \) reflects the excessive prudence in regulation guidelines for investors.
- Certain frameworks, such as the Prudent Person Investment Principles (PPIP) required by EU Solvency II, require the investor to act prudently.
- However, the consideration of small \( p \) may cause standard simulation procedures to break down.
Intuitively, for $L(p) > l$ to happen, at least one of the obligors needs to experience a loss over its threshold, which has probability $p$.

Depending on the value of $l$, it may require multiple obligors to default, which under asymptotically dependent still has a probability of order $p$.

Thus, for the asymptotically dependent case, we expect that as $p \downarrow 0$ the probability $\Pr(L(p) > l)$ decays to 0 at rate $p$. 
An Exploratory Numerical Study

- $d = 5$ obligors
- each loss $X_i$ distributed by Pareto($\alpha_i, \theta_i$)
- $X$ possesses a Gumbel copula

$$C(u) = \exp \left\{ - \left( \sum_{i=1}^{5} (-\ln u_i)^r \right)^{1/r} \right\}$$

- each $G_i$ specified to be a uniform distribution over $(0, y_G)$
- $\alpha = 2$, $\theta_i = i$, and $r = 5$
- $p$ ranges from 0.001 to 0.05
- $l = 0.3$
- $N = 10^6$ simulations
Figure: The decaying rate of the estimated probability $\Pr(L(p) > l)$ as $p$ approaches 0 (with $y_G = 3$ in (a) and $y_G = 10$ and (b)).
Approximating the Distribution of LGD

Theorem (T. and Yuan (2015, working))

Assume that $X \in \text{MRV}_{-\alpha}$ with limit measure $\nu$, and that all the components of $X$ are comparable in the sense that, for some $c_i > 0$,

$$\lim_{x \to \infty} \frac{F_i(x)}{F(x)} = c_i.$$ 

Then it holds for every $l \in (0, 1)$ that

$$\lim_{p \downarrow 0} \frac{\Pr (L(p) > l)}{p} = \nu \left( c^{1/\alpha} B_l \right),$$

where $B_l = \left\{ y \in [0, \infty]^d : \sum_{i=1}^d e_i G_i (y_i - 1) > l \right\}$. 
To numerically check the accuracy of the asymptotic approximation provided by relation (3), we follow the modeling specifications before:

- $\alpha = 1$
- $r = 5$
- $\gamma_G = 2$
- ...
Example 1a: \( d = 3, \ l = 0.1, \ N = 10^5 \)
Example 1b: $d = 3, l = 0.1, N = 10^6$
Example 2a: $d = 3$, $l = 0.3$, $N = 10^5$
Example 2b: $d = 3, I = 0.3, N = 10^6$
Example 3a: $d = 3, \ l = 0.5, \ N = 10^5$
Example 3b: $d = 3$, $l = 0.5$, $N = 10^6$

Asymptotic approximation / empirical estimation

Qihe Tang (University of Iowa)
Losses Given Default with Extreme Risks
June 24, 2015 26 / 31
Example 4a: $d = 5$, $l = 0.1$, $N = 10^6$
Example 4b: \( d = 5, \ l = 0.3, \ N = 10^6 \)
Example 4c: $d = 5$, $l = 0.5$, $N = 10^6$
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- We studied the limit distribution of the LGD as the default probability becomes small (meaning that the portfolio consists of assets of good credit quality).

- In the studies we applied the sophisticated concept of MRV from EVT.

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Thank You for Your Attention!!