6.2 Introduction to Probability

Terms:

"What are the chances of…?"

- Personal probability (subjective)
  - Based on feeling or opinion.
  - Gut reaction.
- Empirical probability (evidence based)
  - Based on experience and observed data.
  - Based on relative frequencies.
- Theoretical probability (formal)
  - Precise meaning.
  - Based on assumptions.
  - In the long run…

The Deal

- Bag o' chips (poker chips).
  - Some are red.
  - Some are white.
  - Some are blue.

- Action: Draw 1 chip from the bag.
  - We don't know how many of each of the **three colors** are in the bag.
  - *Assumption*: Each chip has equal probability of being chosen (Same size, same weight, same texture, etc.)

Possible outcomes:

- Draw a red chip.
- Draw a white chip.
- Draw a blue chip.

Definitions

Outcomes are the most basic possible results of observations or experiments.

An easy way to list the possible outcomes here: R, W, B

Example: Two-coin toss

- Visual of possible outcomes:

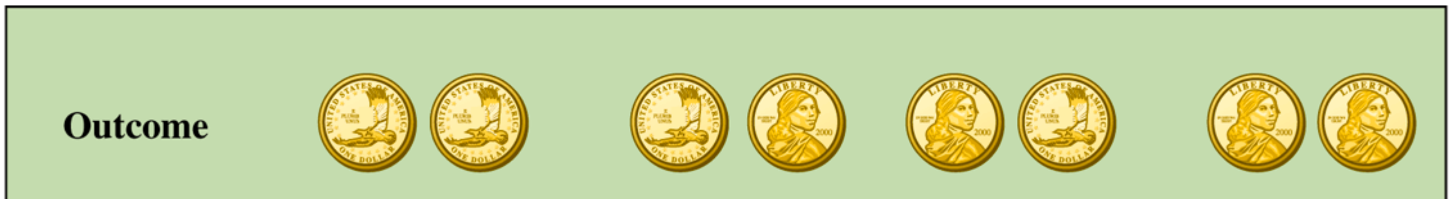
TT TH HT HH

Note: each of these 4 outcomes is equally likely.

- Suppose we're interested in the **number of heads**…

Example: Two-coin toss

- Visual of possible outcomes:



TT TH HT HH

- **Number of heads**…

0 1 1 2

Example: Two-coin toss

- The **event** of getting 2 heads.
  - Occurs only when a HH is tossed.
  - One outcome coincides with this event.
- The **event** of getting 1 head.
  - Occurs when <u>either</u> HT or TH is tossed.
  - Two outcomes coincide with this event.
  - Two different outcomes represent the same event.

Definitions

An **event** is a <u>collection of one or more outcomes</u> that share a property of interest.

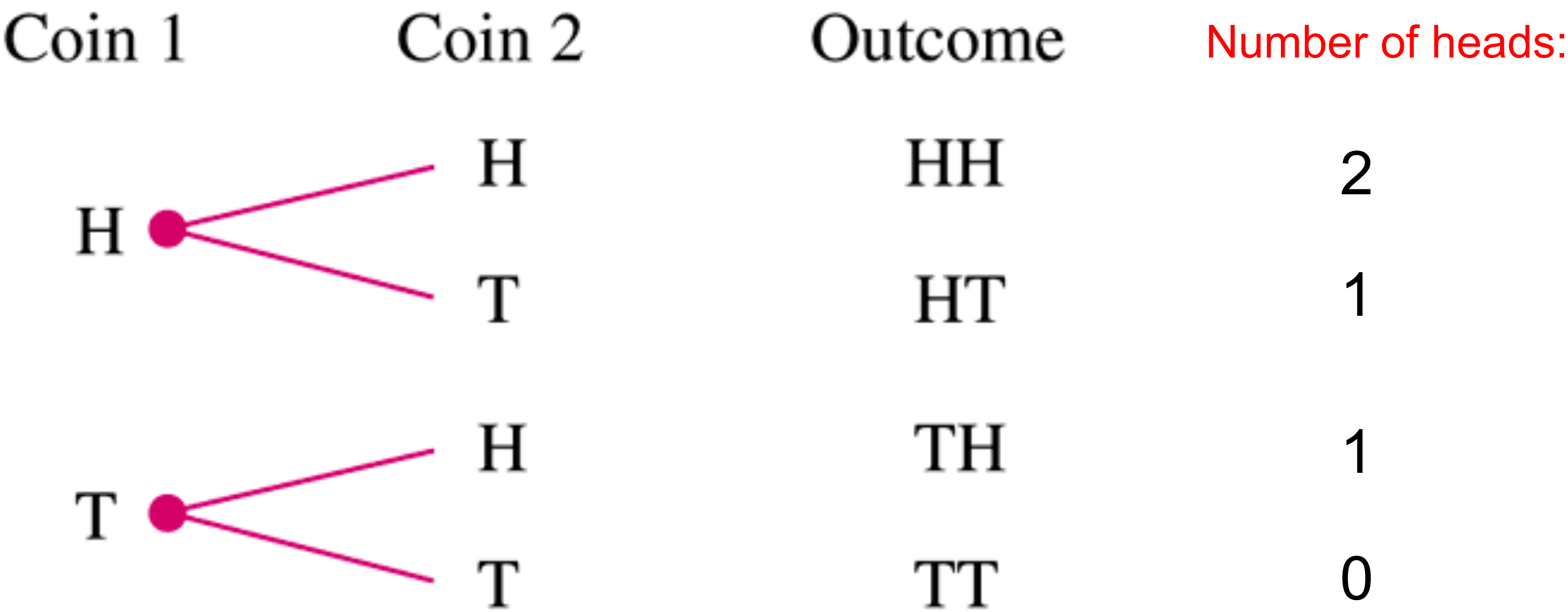
Example: Two-coin toss

- Probability of getting 1 head.
  - P(1 head) = 2/4 = 0.50
- Probability of getting 0 heads.
  - P(0 heads) = 1/4 = 0.25
- Probability of getting 2 head.
  - P(2 heads) = 1/4 = 0.25

Sum to 1.

We MUST get either 0, or 1, or 2 heads on the toss.
So, the probabilities of these 3 events MUST sum to 1.

Example: Two-coin toss



Note: each of these 4 outcomes is equally likely (fair coin), and each has a ¼ chance of occurring.

Probability Rules

- A probability is a number between 0 and 1, inclusive.

$$0 \le P \le 1$$

- A probability of 0 means the the event is impossible and can not happen.
  - P(event)=0.
- A probability of 1 means the event is certain.
  - P(event)=1.

Example: Roll of a 6-sided die

- Six possible **outcomes**: 1, 2, 3, 4, 5, 6
  - The **event** of rolling an odd number on a die.
    - Occurs with a 1, 3, or 5.
    - Three outcomes in this event.
- *P(rolling an odd) = 3/6 = ½ = 0.5*
- *P(rolling a number larger than 6)=0*

Note: each of these 6 outcomes is equally likely, and each has a 1/6 chance of occurring.

Equally likely outcomes

Theoretical Method for Equally Likely Outcomes

Step 1. Count the total number of possible outcomes.

Step 2. Among all the possible outcomes, count the number of ways the event of interest, *A*, can occur.

Step 3. Determine the probability, $P(A)$, from

$$P(A) = \frac{\text{number of ways } A \text{ can occur}}{\text{total number of outcomes}}$$

We used this technique for the events of
'Rolling an odd' (1, 2, 3, 4, 5, 6 equally likely)
and 'Throwing 1 head' (HH, TH, HT, TT equally likely).

Theoretical Probabilities

- Precise meaning
- Based on assumptions
- We might say… "In the long run…"

- Example: For a fair coin, we say

  P(head)=0.5 and P(tail)=0.5

  and this is based on how we *assume* a fair coin behaves.

Theoretical method for equally likely outcomes

Step 1. Count the total number of possible outcomes.

Step 2. Among all the possible outcomes, count the number of ways the event of interest, A, can occur.

Step 3. Determine the probability, $P(A)$, from

$$P(A) = \frac{\text{number of ways } A \text{ can occur}}{\text{total number of outcomes}}$$

Theoretical method for equally likely outcomes

- Example: single coin toss
  - 2 possible equally likely outcomes (H, T)
  - P(head)=1/2

- Example: roll a 6-sided die (1,2,3,4,5,6)
  - 6 possible equally likely outcomes
  - P(1)=1/6
  - P(even)=3/6=1/2

Theoretical method for equally likely outcomes

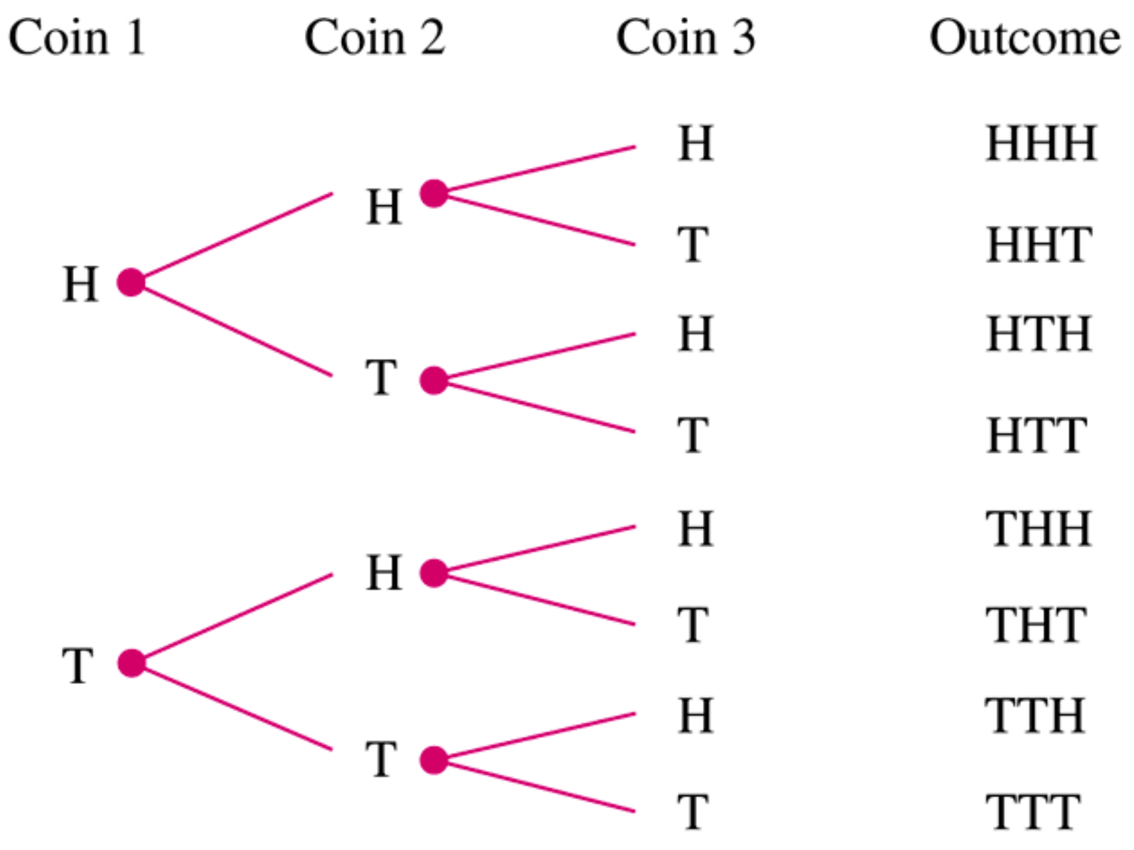
- Example: toss 2 coins
  - 4 possible equally likely outcomes (HH,HT,TH,TT)
  - P(1 head)=2/4=1/2
  - P(2 heads)=1/4
- Example: roll an 8-sided die
  - 8 possible equally likely outcomes
  - P(1)=1/8
  - P(odd)=4/8=1/2

Theoretical method for equally likely outcomes

- Example: lottery with 3-digit number
  - 1000 possible equally likely outcomes (000-999)
  - P(the number 546 chosen)=1/1000
  - P(a number in the 300’s)=100/1000=1/10

- How many equally likely outcomes are there if you toss a coin 3 times?

Theoretical method for equally likely outcomes



(b)

Theoretical method for equally likely outcomes

- How many equally likely outcomes are there if you roll two 6-sided dice?

Theoretical method for equally likely outcomes

- How many equally likely outcomes are there if you roll two 6-sided dice?

Roll number 2

Roll number 1

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

Theoretical method for equally likely outcomes

- Example: Roll two 6-sided dice.
  - P(rolling a sum of 7) = 6/36 = 1/6
  - P(rolling doubles) = 6/36 = 1/6
  - P(rolling a 1 on the first roll) = 6/36 = 1/6
  - P(2 on first roll, 6 on second)=1/36
  - P(rolling a sum of 11) = 2/36 = 1/18

Example:
choosing a pair of letters

- Using Scrabble game tiles, the letters A, B, and C are placed in a bag.
- Two letters are pulled out of the bag simultaneously as (Letter 1, Letter 2).
- Find P(A is chosen in the pair).
- What's the probability that A is left in the bag?

Counting techniques

- Sometimes, instead of writing out *ALL* the outcomes (cumbersome), we instead just consider the *counts* of the number of outcomes for analysis.

- In cases where the operation can be described in a sequence of steps, we can multiply the number of ways each step can be done to find the total number of possible outcomes.

Example: Guitar Hero

- In the game Guitar Hero, you get to choose a character/guitar/venue combination.
- You have 8 characters, 6 guitars, and 4 venues to choose from.
- There are 8 x 6 x 4 = 192 possible outcomes.

Example: Tossing a coin 3 times

- In each toss, there are 2 possible options.
- There are 2 x 2 x 2 = 2^3 = 8 possible outcomes.
- If we wrote them all out, we have:
  HHH, HHT, HTH, THH, HTT, THT, TTH, TTT.

Counting Outcomes

Suppose process A has *a* possible outcomes and process B has *b* possible outcomes. Assuming the outcomes of the processes do not affect each other, the number of different outcomes for the two processes combined is $a \times b$.

This idea extends to any number of processes.

For example, if a third process C has *c* possible outcomes, the number of possible outcomes for the three processes combined is $a \times b \times c$.

Example: Car dealer

- A brand of car comes in five different styles, with four types of engines, and two types of transmission.
  - How many cars would a dealer have to stock if he included one for each style-engine-transmission combination?

Example: Multiple steps

- How many outcomes are there if you roll a fair die and toss a fair coin?
  - Answer: The first process, rolling a fair die, has six outcomes (1, 2, 3, 4, 5, 6). The second process, tossing a fair coin, has two outcomes (H, T). Therefore, there are $6 \times 2 = 12$ outcomes for the two processes together (1H, 1T, 2H, 2T, . . . , 6H, 6T).

12 outcomes

Empirical Probability

(relative frequency probability)

- Based on observed evidence
  - Based on experience and observed data.
  - Based on relative frequencies.
- If there are 15 accidents in a year, we might say that the probability of an accident on a randomly chosen day is 15/365 = 0.041

Empirical Probability

- Example:

Over the last month, my 2-year old has been presented with green beans 20 times, and rejected them 18 times. The *relative frequency probability* that he will reject green beans is 18/20 = 9/10 = 0.90

Empirical Probability

- Example:

Last winter, I drove my car daily in December and January and it failed to start on 8 days. The *relative frequency probability* that the car will not start is $8/62 = 4/31 = 0.129$

Relative Frequency Method

Step 1. Repeat or observe a process many times and count the number of times the event of interest, A, occurs.

Step 2. Estimate $P(A)$ by

$$P(A) = \frac{\text{number of times } A \text{ occurred}}{\text{total number of observations}}$$

Three Approaches to Finding Probability

A **theoretical probability** is based on assuming that all outcomes are equally likely. It is determined by dividing the number of ways an event can occur by the total number of possible outcomes.

A **relative frequency** probability is based on observations or experiments. It is the relative frequency of the event of interest.

A **subjective probability** is an estimate based on experience or intuition (not something we really consider in statistics class).

Probability of an event *not* occurring

- Sometimes it's easier to compute the probability of something NOT occurring.

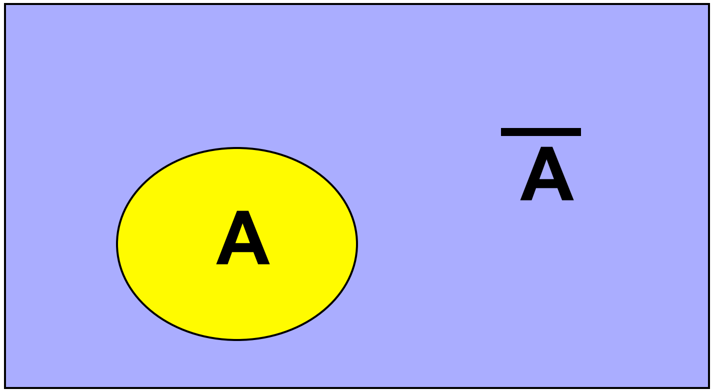
- In this case, the Complement rule can be useful:
  - P(A) + P(not A) = 1
    - P(rolling an Even) +P(not rolling an Even)=1
    - 'A' and 'not A' are complements.
    - It's an either/or situation.

The following two events are complements:

Event 1: Roll an even die

Event 2: Roll an odd die

Probability of an event *not* occurring



- Complement rule
  - P(A) + P(not A) = 1
  - P(A) = 1 – P(not A)
  - The probability that an event occurs is 1 minus the probability that it does not occur.
  - The event *not A* is called the complement of A.

Symbol for "complement of A" is $\overline{A}$

  - What is the probability of rolling a 6-sided die and getting a value 2 or larger?
    - P(2 or larger)=1-P(not 2 or larger)=1-1/6=5/6

Complements

- Consider rolling two 6-sided dice.
  - P(rolling a sum of 7) = 6/36=1/6
  - P(not rolling a sum of 7) = 1 - 1/6 = 5/6
  - Or P(not rolling a sum of 7) = 30/36 = 5/6

Roll number 2

Roll number 1

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
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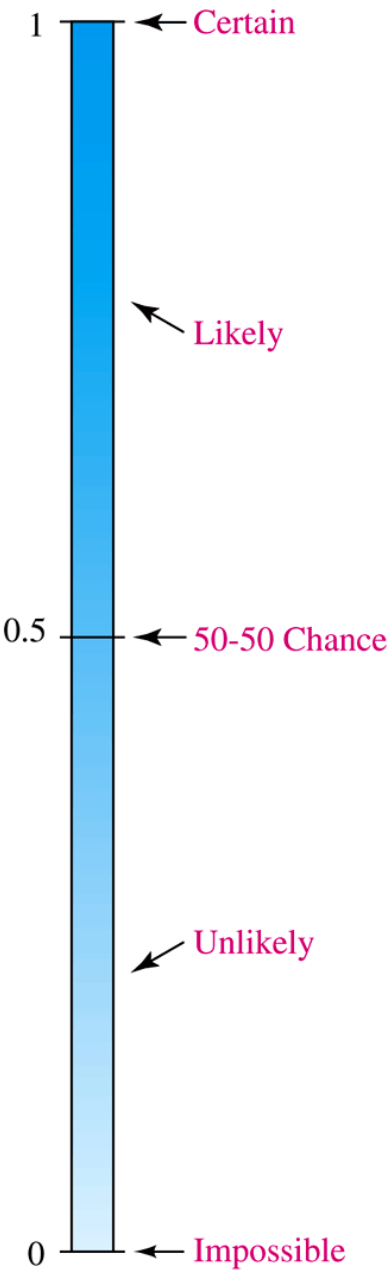
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

Expressing Probability

The probability of an event, expressed as *P*(event), is always between 0 and 1 inclusive.

A probability of 0 means that the event is impossible, and a probability of 1 means that the event is certain.



The scale shows various degrees of certainty as expressed by probabilities.