STA 611: Introduction to Mathematical Statistics – Fall 2014
Homework 8 – Due November 6, 2014

These exercises are meant to be representative of the material in Chapters 8.1-8.5 in DeGroot and Schervish.

1. Let $X_1, \dots, X_n$ be a random sample from $N(\mu, 9)$. In each case, determine how large n must be for the statement to hold

   (a) $E((\overline{X}_n - \mu)^2) \leq 0.1$

   (b) $P(|\overline{X}_n - \mu| \leq 0.1) \geq 0.95$

2. Let $X_i \sim N(i, i^2)$ be independent for $i = 1, 2, 3$. Use the X_i's to construct a statistic that has the following distributions:

   (a) χ^2_3

   (b) t_2

3. The t_m distribution was introduced as a ratio

$$T = \frac{Z}{\sqrt{Y/m}}$$

   of independent random variables $Z \sim N(0, 1)$ and $Y \sim \chi^2_m$. Show that T^2 is a ratio of two independent Gamma-distributed random variables, each with mean one. Find the parameters for each Gamma distribution.

4. Let $X_1, \dots, X_n$ be a random sample from $N(\mu, \sigma^2)$, where both μ and σ^2 are unknown.

   (a) For $n = 25$ calculate the following probabilities

      i. $P\left(0.8\sigma^2 \leq \frac{1}{n}\sum_{i=1}^n (X_i - \mu)^2 \leq 1.2\sigma^2\right)$

      ii. $P\left(0.8\sigma^2 \leq \frac{1}{n}\sum_{i=1}^n (X_i - \overline{X})^2 \leq 1.2\sigma^2\right)$

   (b) Find the smallest n so that $P\left(\frac{1}{n\sigma^2}\sum_{i=1}^n (X_i - \overline{X})^2 \leq 1.4\right) \geq 0.9$

5. Suppose you shoot an arrow at a very large circular target. Suppose the center of the target is at the origin and let (X, Y) be the coordinates of the point of impact. Assume that X and Y are independent $N(0, 1)$ random variables.

   (a) What is the probability that the point of impact is no further than 2 units from the origin? (*Hint: It may be easier to work with the distribution of the squared distance.*)

6. The following measurements were made:

$$0.95, 0.85, 0.92, 0.95, 0.93, 0.86, 1.00, 0.92, 0.85, 0.81$$
$$0.78, 0.93, 0.93, 1.05, 0.93, 1.06, 1.06, 0.96, 0.81, 0.96$$

A histogram of this data suggests that the distribution is approximately $N(\mu, \sigma^2)$ with unknown μ and σ^2.

(a) Calculate the observed two-sided (symmetric) 95% confidence interval for μ.

(b) Calculate the observed two-sided (symmetric) 95% confidence interval for σ^2.

(c) Calculate the observed two-sided (symmetric) 95% confidence interval for σ.

7. Let $X_1, \ldots, X_n$ be a random sample from Uniform$(0, \theta)$

(a) Let $Y = \max(X_1, \ldots, X_n)$. Show that Y/θ is a pivotal quantity

(b) Construct a $100\gamma\%$ confidence interval for θ