Chapter 1 sections

- 1.4 Set Theory
    - SKIP: Real number uncountability
- 1.5 Definition of Probability
- 1.6 Finite Sample Spaces
- 1.7 Counting Methods
- 1.8 Combinatorial Methods
- 1.9 Multinomial Coefficients
- SKIP: 1.10 The Probability of a Union of Events
- SKIP: 1.11 Statistical Swindles

Announcements

- First homework due Thurs. Sep 4
- Due in class or in my mail box (211 Old Chemistry building) **by 5pm Thursday**
- Tentative lecture schedule on the website

Definition of Probability

Def: Probability measure

A probability on a sample space S is a function $P(A)$ for all events A that satisfies Axioms 1, 2 and 3

Axiom 1 $P(A) \geq 0$ for all events A

Axiom 2 $P(S) = 1$

Axiom 3 For every infinite sequence of disjoint events $A_1, A_2, A_3, \ldots,$

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

Finite sequence of disjoint events: It follows from Axiom 3 that

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} (A_i)$$

Definition of Probability

- The axioms of probability are properties that we intuitively expect a probability to have
- The axioms are not concerned with the different interpretations of what probability means
- All probability theory is built on these axioms
- The mathematical foundations of probability (including these axioms) were laid out by Andrey Kolmogorov in 1933

Examples of probability measures

Tossing one fair coin

- $S = \{H, T\}$
- Since the coin is fair we set $P(H) = P(T) = 1/2$
- All axioms are satisfied

Random point in the unit square

$S = \{(x, y) : 0 \leq x \leq 1,\ 0 \leq y \leq 1\}$

- For an event $A \subset S$ we set $P(A) =$ the area of A
- Obviously $P(S) = 1$ and $P(A) \geq 0$ for all A
- Axiom 3: For any sequence of disjoint subsets in S the area of all of them is the same as the sum of the areas of each one.

What is the probability of $(0.1, 0.3)$?

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What is the probability of $(0.1, 0.3)$? Area of a point is zero so $P((x, y) = (0.1, 0.3)) = 0$. For continuous sample spaces, a zero probability does not necessarily mean impossibility.

Properties of probability

Theorem

If P is a probability and A and B are events then

- *If $A \subset B$ then $P(A) \leq P(B)$*
- $P(A \cap B^c) = P(A) - P(A \cap B)$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

These can be shown from the three axioms of probability.

Theorem

If P is a probability and A is an event then

- $P(A^c) = 1 - P(A)$
- $P(\emptyset) = 0$
- $0 \leq P(A) \leq 1$

Useful inequalities

Theorem: Bonferroni inequality

For any events $A_1, A_2, \ldots, A_n$

$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i) \quad \text{and} \quad P\left(\bigcap_{i=1}^{n} A_i\right) \geq 1 - \sum_{i=1}^{n} P(A^c)$$

The second inequality can be derive from the first one.

Example 1.5.11

A point (x, y) is to be selected from the square S containing

- S: The square $\{(x, y) : 0 \leq x \leq 1,\ 0 \leq y \leq 1\}$
- Probability of a subset of S is equal to the area of that subset

Find the probabilities of the following subsets:

1. The subset of points such that $(x - 1/2)^2 + (y - 1/2)^2 \geq 1/4$
2. The subset of points such that $1/2 < x + y < 3/2$
3. The subset of points such that $y \leq 1 - x^2$
4. The subset of points such that $x = y$

Terminology

- **Finite vs. Infinite**: The number of people in this room is finite. The amount of time it takes for everyone to love statistics may be infinite :-).
- **Discrete vs. continuous**: Time moves continuously but the hours on a digital clock move discretely.
- **Countable vs. uncountable**: $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ are countable, $\mathbb{R}$ is uncountable
- **Simple sample space**: A finite sample space, $S = \{s_1, s_2, \ldots, s_n\}$, where every outcome is equally likely, i.e.

$$P(s_i) = \frac{1}{n} \text{ for all } i \quad \text{and} \quad P(E) = \frac{\#E}{n}$$

  - Need “only” count the outcomes in an event to find it’s probability

Counting

- What are the chances of winning the lottery?
- In a hand of 5 cards, how likely is it to get four aces?

Multiplication rule

If a job consists of k parts ($k \geq 2$), and the ith part has n_i possible outcomes regardless of what outcomes came before, then the job can be done in

$$n_1 \times n_2 \times \cdots \times n_k \quad \text{ways}$$

- For example: A frozen yogurt parlor has three types of frozen yogurts.
  If you also get one topping (sprinkles, mini m&m's or skittles) and a sause (chocolate or caramel) there are $3 \times 3 \times 2 = 18$ ways you can enjoy the frozen yogurt.

Counting

Permutations and combinations

- In how many ways can people in a committee of 4 be assigned four roles (president, vice president, treasurer, secretary)? This is **permutation**

$$\text{In} \quad P_{4,4} = 4! = 24 \quad \text{ways}$$

- In how many ways can 10 people form a committee of 4? This is **combination** (think: subset)

$$\text{In} \quad C_{10,4} = \binom{10}{4} = \frac{10!}{(10-4)!4!} = 210 \quad \text{ways}$$

Counting - Overview

Number of ways one can pick k things out of n depends the situation

- Ordered samples with replacement

$$n^k$$

- Ordered samples without replacement (Permutation if $n = k$):

$$P_{n,k} = n \times (n-1) \times \cdots \times (n-k+1) = \frac{n!}{(n-k)!}$$

- Unordered samples with replacement:

$$\binom{n+k-1}{k}$$

- Unordered samples without replacement (Combinations):

$$C_{n,k} = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Examples

Using counting to find probabilities

1. You are in a room of 12 people. What is the probability that at least 2 of those people have the same birthday?
2. What is the probability that in a hand of 5 cards you get four aces?

Binomial and Multinomial theorems

- Binomial coefficient - number of ways to choose k items out of n (without replacement):

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

- Binomial Theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

- Multinomial coefficient - number of ways to divide n items into k different groups:

$$\binom{n}{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1!n_2!\cdots n_k!}, \qquad \begin{array}{l} n_j = \text{no. of items in group } j \\ \text{and } n_1 + n_2 + \cdots + n_k = n \end{array}$$

- Multinomial Theorem:

$$(x_1 + x_2 + \cdots + x_k)^n = \sum_{n_1+\cdots+n_k=n} \binom{n}{n_1, n_2, \ldots, n_k} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}$$

END OF CHAPTER 1

Chapter 2

- 2.1 The Definition of Conditional Probability
- 2.2 Independent Events
- 2.3 Bayes' Theorem
- SKIP 2.4 The Gambler's Ruin Problem

Definition of conditional probability

If we know that event B has occurred what is the probability of event A?

Def: Conditional Probability

The conditional probability of event A given that event B has occurred:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad \text{if } P(B) > 0$$

- Conditional probabilities behave just like probabilities - we can show that they satisfy the three Kolmogorov axioms.
- $\Rightarrow$ All theorems stated about probabilities also hold for conditional probabilities!
  - For example: $P(A|B) = 1 - P(A^c|B)$

Law of total probability

Def: Partition

Let S be a sample space. If $A_1, A_2, A_3, \dots$ are disjoint and $\bigcup_{i=1}^{\infty} A_i = S$ then the collection $A_1, A_2, A_3, \dots$ is called a partition of S

Law of total probability

If events $B_1, \dots, B_k$ form a partition of the sample space S and $P(B_j) > 0$ for all j, then for every event A in S

$$P(A) = \sum_{i=1}^{k} P(B_j)P(A|B_j)$$

Monty Hall Problem

- Suppose you're on a Monty Hall's game show, and you're given the choice of three doors.
- Behind one door is a car; behind the others, goats. You win the price behind the door your choose.
- You pick door No. 1 (but the door is not opened), and the host, who knows what's behind the doors opens door No. 3 which has a goat. (Monty will never show you the car)

Monty Hall then says to you:

"Do you want to stay with door No.1 or switch to door No. 2?"

For an alternative illustration, using a different argument see http://www.youtube.com/watch?v=mhlc7peGlGg&feature=related

Bayes' Theorem

Bayes' Theorem

If $B_1, B_2, \ldots, B_k$ form a partition of S and $P(B_j) > 0\ \forall j$ and $P(A) > 0$ then

$$P(B_i|A) = \frac{P(B_i)P(A|B_i)}{\sum_{j=1}^{k} P(B_j)P(A|B_j)}$$

Example: Blood test for a disease

- $P(+|\text{sick}) = 0.9$ and $P(-|\text{healthy}) = 0.85$
- Prevalence of the disease in the country is 5%
- If you get a negative test result, what is the prob. that you are sick?

$$\begin{aligned} P(\text{sick}|-) &= \frac{P(\text{sick})P(-|\text{sick})}{P(\text{sick})P(-|\text{sick}) + P(\text{healthy})P(-|\text{healthy})} \\ &= \frac{0.05 \times 0.1}{0.05 \times 0.1 + 0.95 \times 0.85} = 0.00615 \end{aligned}$$

Independence

Def: Independent events

Two events A and B are said to be (statistically) independent if

$$P(A \cap B) = P(A) \times P(B)$$

- Consequence: If A and B are independent then

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A) \quad \text{and} \quad P(B|A) = P(B)$$

Theorem

If A and B are independent, so are the following pairs

- A and B^c
- A^c and B
- A^c and B^c

Mutually independent events

Def: Mutually independent events

Events $A_1, A_2, \ldots, A_k$ are mutually independent if for every subset $A_{i_1}, \ldots, A_{i_j}$, $j = 2, \ldots, k$

$$P(A_{i_1} \cap \cdots \cap A_{i_j}) = P(A_{i_1}) \times \cdots \times P(A_{i_j})$$

Note:

- $P(A \cap B \cap C) = P(A)P(B)P(C)$ does not imply mutual independence.
- Pairwise independence does not imply mutual independence.

Def: Conditional independence

Events $A_1, A_2, \ldots, A_k$ are conditionally independent given B if for every subset $A_{i_1}, \ldots, A_{i_j}$, $j = 2, \ldots, k$

$$P(A_{i_1} \cap \cdots \cap A_{i_j} | B) = P(A_{i_1} | B) \times \cdots \times P(A_{i_j} | B)$$

Example: Tossing two dice

$P(A \cap B \cap C) = P(A)P(B)P(C)$ does not imply mutual independence

Consider the following three events:

- Event A: Get doubles. $P(A) = 6/36 = 1/6$
- Event $B = \{7 \leq \text{ sum } \leq 10\}$. $P(B) = 18/36 = 1/2$
- Event $C = \{\text{the sum is } 2, 7 \text{ or } 8\}$. $P(C) = 12/36 = 1/3$

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We have $P(A \cap B \cap C) = P(A)P(B)P(C)$

- $A \cap B \cap C = \{(4,4)\}$ so $P(A \cap B \cap C) = 1/36$ and $P(A)P(B)P(C) = 1/6 \times 1/2 \times 1/3 = 1/36$

Example: Tossing two dice

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We have $P(A \cap B \cap C) = P(A)P(B)P(C)$

- $A \cap B \cap C = \{(4, 4)\}$ so $P(A \cap B \cap C) = 1/36$ and $P(A)P(B)P(C) = 1/6 \times 1/2 \times 1/3 = 1/36$

But $P(A \cap B) \neq P(B)P(C)$ since

- $B \cap C = \{\text{sum is 7 or 8}\}$ so $P(B \cap C) = 11/36$ but $P(B)P(C) = 1/2 \times 1/3 = 1/6$

Therefore, A, B and C are NOT mutually independent.

Example: Letters

Events can be pairwise independent without being mutually independent

Let S be a simple sample space where

$$S = \{\mathrm{aaa}, \mathrm{bbb}, \mathrm{ccc}, \mathrm{abc}, \mathrm{acb}, \mathrm{bac}, \mathrm{bca}, \mathrm{cab}, \mathrm{cba}\}$$

- Let $A_i = \{i\text{th place has an a }\}$.
- Easy to see that $P(A_i) = \frac{3}{9} = \frac{1}{3}$ for $i = 1, 2, 3$
- Only one element has a in two places so

$$P(A_1 \cap A_2) = P(A_1 \cap A_3) = P(A_2 \cap A_3) = \frac{1}{9}$$

So A_1, A_2 and A_3 are *pairwise* independent.

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- Only one element has a in two places so

$$P(A_1 \cap A_2) = P(A_1 \cap A_3) = P(A_2 \cap A_3) = \frac{1}{9}$$

So A_1, A_2 and A_3 are *pairwise* independent.

- $P(A_1 \cap A_2 \cap A_3) = \frac{1}{9}$ but $P(A_1)P(A_2)P(A_3) = \frac{1}{3^3} = \frac{1}{27}$

So A_1, A_2 and A_3 are NOT mutually independent.

END OF CHAPTER 2