Chapter 3 sections

- 3.1 Random Variables and Discrete Distributions
- 3.2 Continuous Distributions
- 3.3 The Cumulative Distribution Function
- 3.4 Bivariate Distributions
- 3.5 Marginal Distributions
- 3.6 Conditional Distributions
- Just skim: 3.7 Multivariate Distributions (generalization of bivariate)
- 3.8 Functions of a Random Variable
- 3.9 Functions of Two or More Random Variables
  - SKIP: pages 180 -186
- SKIP: 3.10 Markov Chains

Transformations of Random Variables

- If X is a random variable then any function of X, $g(X)$, is also a random variable
- Sometimes we are interested in $Y = g(X)$ and need the distribution of Y
  - Example: Say we have the distribution of the service rate X, then what is the distribution of the average waiting time $Y = 1/X$?
- We can use the distribution of X to get the distribution of Y:

$$P(Y \in A) = P(g(X) \in A)$$

$$\text{e.g.}\quad P(Y \leq y) = P(1/X \leq y) = P(X \geq 1/y)$$

Depending on the function g we can sometimes obtain a tractable expression for the probability of Y

Transformations of Random Variables

Inverse mapping

Let $g(x) : \mathcal{X} \to \mathcal{Y}$. The *inverse mapping* is defined as

$$g^{-1}(A) = \{x \in \mathcal{X} : g(x) \in A\}$$

For a set of one point we write

$$g^{-1}(\{y\}) = g^{-1}(y) = \{x \in \mathcal{X} : g(x) = y\}$$

We can therefore write:

$$\begin{aligned} P(Y \in A) &= P(g(X) \in A) = P(\{x \in \mathcal{X} : g(x) \in A\}) \\ &= P(X \in g^{-1}(A)) \end{aligned}$$

Transformation of a discrete random variable

- If $Y = g(X)$ where X is a discrete r.v. with support $\mathcal{X}$ then Y is also a discrete r.v. and

$$f_Y(y) = P(Y = y) = P(X \in g^{-1}(y)) \\ = \sum_{x \in g^{-1}(y)} P(X = x) = \sum_{x \in g^{-1}(y)} f_X(x)$$

for all $y \in \mathcal{Y} = \{y : y = g(x), x \in \mathcal{X}\}$

Example:

- Let $X \sim \mathrm{Binomial}(n, p)$, i.e.

$$f_X(x) = P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, 2, \ldots, n$$

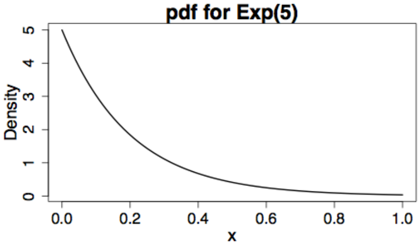
- What is the pf of $Y = n - X$ (i.e. the number of failures)?

Example 1: Exponential distribution (Continuous r.v.)

Let X be a random variable with pdf

$$f_X(x) = \lambda \exp(-\lambda x), \quad x > 0$$

where λ is a positive constant



We say that X is *exponentially distributed with parameter (rate)* λ or $X \sim \mathrm{Exp}(\lambda)$.

1. What is the cdf of X?
2. What is the distribution of $Y = \alpha X$ where α is a positive constant?
3. What is the distribution of $W = X^2$?

Example 2: Double exponential distribution

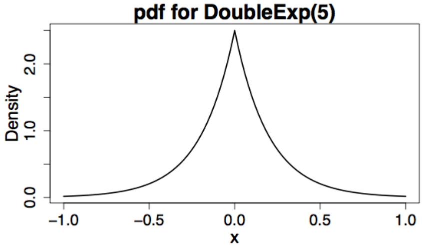
Also called Laplace distribution

Let X be a random variable with pdf

$$f_X(x) = \frac{\lambda}{2}\exp(-\lambda|x|), \quad x \in \mathbb{R}$$

where λ is a positive constant

- What is the cdf of X?
- What is the cdf of $W = X^2$?



Example 3: cdf transformation

Again we consider the exponential distribution.

Let $X \sim \mathrm{Exp}(\lambda)$, then X has the pdf

$$f_X(x) = \lambda e^{-\lambda x}, \quad x > 0$$

where λ is a positive constant.

(a) Let $F_X(x)$ be the cdf found in Ex. 1. Find the distribution of $Y = F_X(X)$.

(b) Find the inverse cdf F_X^{-1} and the distribution of $F_X^{-1}(U)$ where $U \sim \mathrm{Uniform}(0,1)$.

Probability integral transformation

Theorem

1. Let X have a continuous cdf F and let $Y = F(X)$. Then $F(X) \sim \mathrm{Uniform}(0, 1)$.
2. Let $Y \sim \mathrm{Uniform}(0, 1)$ and let F be a continuous cdf with quantile function F^{-1}. Then $X = F^{-1}(Y)$ has cdf F.

- This theorem is useful when we want to generate random numbers from some distribution.
- If F^{-1} is available in closed form we can simply generate uniform random numbers and then transform them using F^{-1}.
- Therefore, much of the effort concerning generating (pseudo) random numbers has been concentrated on generating uniform random numbers.

One-to-one functions

Consider function $y = g(x)$, $g : \mathcal{X} \to \mathcal{Y}$

- $g(x)$ is *one-to-one* if each $y \in \mathcal{Y}$ corresponds to exactly one $x \in \mathcal{X}$ (and vice versa)
- $g(x)$ is *onto* if for every $y \in \mathcal{Y}$ there exists an $x \in \mathcal{X}$ such that $y = g(x)$.
- If $g(x)$ is both one-to-one and onto, $g^{-1}(y)$ exists.
- Monotone functions (strictly increasing or strictly decreasing) are both one-to-one and onto

In examples 1 and 2:

- Ex. 1: $Y = \alpha X$ is one-to-one
- Ex. 1: Usually $W = X^2$ is not one-to-one, but since we know X is positive ($\mathcal{X} = (0, \infty)$) it is one-to-one.
- Ex. 2: There X is not restricted to be positive so the function is not one-to-one.

Monotone transformations of continuous r.v.'s

Theorem

Let X be a random variable with pdf $f_X(x)$ and support $\mathcal{X}$ and let $Y = g(X)$ where g is a monotone function.

Suppose $f_X(x)$ is continuous on $\mathcal{X}$ and that $g^{-1}(y)$ has a continuous derivative on $\mathcal{Y} = \{y : y = g(x), x \in \mathcal{X}\}$.

Then the pdf of Y is

$$f_Y(y) = \begin{cases} f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| & \text{if } y \in \mathcal{Y} \\ 0 & \text{otherwise} \end{cases}$$

Example 4: Transformation of the Gamma distribution

Consider the *Gamma distribution with parameters n and β*

Let $X \sim \mathrm{Gamma}(n, \beta)$. Then X has the pdf

$$f(x) = \frac{1}{(n-1)!\beta^n} x^{n-1} e^{-x/\beta}$$

What is the pdf of $Y = 1/X$?

The distribution of Y is called the *Inverse gamma distribution*.

Linear functions

A straightforward corollary:

Linear function

Let X be a random variable with pdf $f_X(x)$ and let $Y = aX + b$, $a \neq 0$. Then

$$f_Y(y) = \frac{1}{|a|} f\left(\frac{y-b}{a}\right)$$

Example: Let X have the pdf

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

This is the pdf of the *Normal distribution* with parameters μ (mean) and σ^2 (variance). Notation: $X \sim N(\mu, \sigma^2)$.

Find the pdf of $Y = \frac{X-\mu}{\sigma}$

Sum of two random variables – Convolution

What is the distribution of $Z = X + Y$?

- If X and Y discrete random variables we get

$$P(Z = z) = \sum_i P(X = i, Y = z - i)$$

$$P(Z = z) = \sum_i P(X = i)P(Y = z - i) \quad \text{if } X \text{ and } Y \text{ are independent}$$

- X and Y discrete random variables:

$$f_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}(t, z - t)dt$$

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(t)f_Y(z - t)dt \quad \text{if } X \text{ and } Y \text{ are independent}$$

this is called the *convolution formula*.

Example: What is the distribution of $X + Y$ for independent standard normals X and Y?

END OF CHAPTER 3