1. The probability density function of (X, Y) is defined by

$$f_{X,Y}(x, y) = C(x + 2y), \quad 0 < y < 1, \ 0 < x < 2,$$

and is 0 otherwise.

   a. Find the value of C.
   b. Find the marginal distribution of $f_X(x)$.
   c. Find $E(X)$ and $E(XY)$.
   d. Find $f_{Y|X}(y|x)$.
   e. Find $E(Y|X = x)$.
   f. Are X and Y independent? Why or why not?

2. Let X_1 and X_2 be independent random variables with probability density function $f_X(x) = \lambda \exp(-\lambda x), \quad x > 0$.

   a. Show that the Moment Generating Function of X_1 is

$$E\left(\exp(X_1 t)\right) = \frac{1}{1 - t/\lambda}.$$

   b. Show that the Moment Generating Function of $Z = X_1 + X_2$ is

$$E\left(\exp(Zt)\right) = \left(\frac{1}{1 - t/\lambda}\right)^2.$$

   c. Find $E(Z)$ and $Var(Z)$.

3. You observe data X from a Binomial distribution $X \sim Bin(n, \theta)$, with known n, such that

$$P(X = x|n, \theta) = \binom{n}{x} \theta^x (1 - \theta)^{n-x},$$

and you are interested in the parameter θ. The prior distribution of the parameter θ is Beta distributed with probability density function

$$p(\theta) = \frac{1}{B(\alpha, \beta)} \theta^{\alpha-1} (1 - \theta)^{\beta-1}.$$

Show that the posterior distribution of θ will also have Beta form, and use the formula sheet to find its mean and variance. Explain what happens to the posterior mean (a) as n get large and (b) when α and β are large compared to n.

4. You are preparing for Christmas and have bought 20ft of wrapping paper, and are planning to wrap 10 presents. The average gift requires 2ft of wrapping paper with a standard deviation of 0.5. Use the Central Limit Theorem in order to write down the probability that you will run out of wrapping paper in terms of a standard normal distribution.