Sample size for confidence intervals
with \( \sigma \) known

Lecture 13
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The margin of error

- The **margin of error** is the value that we add onto \( \bar{x} \) and subtract from \( \bar{x} \) to get the endpoints of a confidence interval.
- For confidence intervals for the mean of a normal population with \( \sigma \) known, this is
  \[
  m = z^* \frac{\sigma}{\sqrt{n}}
  \]
  - Equivalently, the margin of error is one half the width of the c.i.
- The margin of error depends on
  - the level of confidence desired
  - the population standard deviation (which we can’t control!)
  - the sample size (not the population size)

Sample size for a study involving a confidence interval

- Suppose a group of obstetricians wish to carry out a study to estimate \( \mu \), the mean birthweight in the population of infants born at UIHC.
- Suppose the obstetricians believe that the population standard deviation of birthweights of infants born at UIHC is the same as that of infants overall in the US.
  \[
  \sigma = 15 \text{ oz}
  \]
- The obstetricians would like a 95% confidence interval for \( \mu \) that is no wider than 4 ounces. That is, they want a margin of error \( \leq 2 \) ounce.
- How many infants do they need in their study?

Let \( m \) denote the margin of error. Then

\[
\begin{align*}
  m &= z^* \frac{\sigma}{\sqrt{n}} \\
  \sqrt{n} &= z^* \frac{\sigma}{m} \\
  n &= \left( \frac{z^* \sigma}{m} \right)^2 \\
  &= \left( \frac{1.96 \cdot 15}{2} \right)^2 \\
  &= 216.09
\end{align*}
\]
- A sample size must always be rounded up, so they need 217 infants in their study.
Sample size continued

What makes a sample size large?

\[ n = \left( \frac{z^* \sigma}{m} \right)^2 \]

Caveats regarding our formula for computing confidence intervals for population means

- The data must be a simple random sample from the population.
  - We are not in too big trouble if the data can plausibly be thought of as observations taken at random from the population.
- “There is no correct method for inference from data haphazardly collected with bias of unknown size. Fancy formulas cannot rescue badly produced data.”*
- Watch out for outliers in your dataset, because they can have a large effect on both the point estimate of \( \mu \) and the confidence interval.

If outliers are not data errors, and if there is no subject-matter reason for deleting them,

get help from a statistician on computing measures of center and intervals that are not sensitive to outliers.

- Check your data for skewness and other signs that the population they came from may not be normal. If the sample size is large (i.e. \( n \geq 30 \)) the central limit theorem says the approach is valid. If the sample size is small, the confidence level will not be correct.
- The formula \( \bar{x} \pm z^* \frac{\sigma}{\sqrt{n}} \) requires that we know the exact value of the population standard deviation \( \sigma \), which we never do.

What to do when we believe the population is normal but we don’t know \( \sigma \)

Assumptions behind this method

- The data are a simple random sample from the population of interest.
- Values in the population follow a normal distribution with mean \( \mu \) and standard deviation \( \sigma \). Both \( \mu \) and \( \sigma \) are unknown.

The sample mean \( \bar{x} \) is still our point estimate of the unknown population mean \( \mu \).

\( \bar{x} \) still comes from a normal distribution with mean \( \mu \) and standard deviation \( \frac{\sigma}{\sqrt{n}} \).

• We will estimate \( \sigma \) by the sample standard deviation \( s \).

• Then we estimate the standard deviation of \( \bar{x} \) by \( \frac{s}{\sqrt{n}} \).

Standard errors

When we use the data to estimate the standard deviation of a statistic, the result is called the standard error of the statistic.

The standard error of the sample mean \( \bar{x} \) is \( \frac{s}{\sqrt{n}} \).

When we claimed to know \( \sigma \), we computed confidence intervals for \( \mu \) as

\[
\bar{x} \pm z^* \frac{\sigma}{\sqrt{n}}
\]

where \( z^* \) was the appropriate cutoff value from a standard normal distribution.

When we don’t know \( \sigma \), we will compute confidence intervals for \( \mu \) as

\[
\bar{x} \pm t^* \frac{s}{\sqrt{n}}
\]

When we are estimating \( \sigma \) with \( s \), we need to make our confidence interval wider to account for the uncertainty in estimation.

• (What if we had gotten a sample that happened to give a sample standard deviation \( s \) that was much smaller than the population standard deviation \( \sigma \)?)

• We do this by multiplying \( \frac{s}{\sqrt{n}} \) by something bigger than \( z^* \).

The \( t \) distribution

• There is a different \( t \) distribution for every sample size.
  – We identify different \( t \) distributions by their degrees of freedom, \( n - 1 \).

• The density curve for \( t \) distributions is
  – symmetric around 0
  – bell-shaped (and has only one mode)

• The spread of \( t \) distributions is greater than the spread of the standard normal distribution.
  – The smaller the degrees of freedom, the more spread out the \( t \) distribution is.
  – The larger the degrees of freedom, the closer the density curve for a \( t \) distribution is to a standard normal curve.
More on the $t$ distribution

If $\bar{x}$ is the sample mean of a simple random sample of size $n$ value from a normal population with mean $\mu$ and standard deviation $\sigma$, then the random quantity

$$ t = \frac{\bar{x} - \mu}{s/\sqrt{n}} $$

follows a $t$ distribution.

Constructing confidence intervals for $\mu$ when $\sigma$ is unknown

To construct a level $C$ confidence interval for $\mu$:

- Draw a simple random sample of size $n$ from the population. The population is assumed to be normal.
- Compute the sample mean $\bar{x}$ and the sample standard deviation $s$.
- Then the level $C$ confidence interval for $\mu$ is

$$ \bar{x} \pm t^* \frac{s}{\sqrt{n}} $$

where $t^*$ cuts off the upper $\frac{1-C}{2}$ area under the density curve for a $t$ distribution with $n - 1$ degrees of freedom.
- Use Table A.2 at the back of your textbook to find $t^*$.

Example

- We have data on a simple random sample of 10 birthweights of infants born at Boston City Hospital.
- We wish to estimate the mean $\mu$ of birthweights in the population served by this hospital.
- This population may be different from the population of all US birthweights, so we cannot assume that we know either $\mu$ or $\sigma$.

<table>
<thead>
<tr>
<th>Infant</th>
<th>Birthweight in ounces</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>97</td>
</tr>
<tr>
<td>2</td>
<td>117</td>
</tr>
<tr>
<td>3</td>
<td>140</td>
</tr>
<tr>
<td>4</td>
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<tr>
<td>5</td>
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<td>6</td>
<td>148</td>
</tr>
<tr>
<td>7</td>
<td>108</td>
</tr>
<tr>
<td>8</td>
<td>135</td>
</tr>
<tr>
<td>9</td>
<td>126</td>
</tr>
<tr>
<td>10</td>
<td>121</td>
</tr>
</tbody>
</table>
First calculate
\[ \bar{x} = 116.90 \quad s = 21.70 \]

The degrees of freedom are 10 - 1 = 9. For a 95% confidence interval, we need the value of \( t^* \) that cuts off an area of .025 in the upper tail.

From Table C, we find \( t^* = 2.262 \).

Our confidence interval is
\[ \bar{x} \pm t^* \frac{s}{\sqrt{n}} = \]
\[ 116.90 \pm 2.262 \frac{21.70}{\sqrt{10}} = \]
\[ 116.90 \pm 15.22 = (101.68, 132.12) \]

The interval is so wide because of
• the relatively small sample size
• the relatively large variation between birth-weights (large \( s \))