Why do we want to study probability

• So far we have studied *descriptive statistics*: methods of describing or summarizing a *sample*.

• We want to move ahead to *inference statistics*: methods for using the data in a sample to draw conclusions about the *population* from which the sample is drawn.

• Methods of inference statistics are based on the question “How often would this method give a correct answer if I used it very, very many times?”

• The laws of probability relate to this question.

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Parameters and statistics

• A *parameter* is a numeric quantity that describes a characteristic of a population.
  
  – We almost never can know the exact value of a parameter, because we would have to measure every member of the population.
  
  – Example: We would like to know the average percent body fat of all Chinese males aged 21 - 65 years.
  
  – We generally use Greek letters to refer to population parameters.
  
  – $\mu$ is the standard symbol for a population mean.

• A *statistic* is a numeric value that can be computed directly from sample data.
  
  – Example: we draw a sample of 10 Chinese males aged 21-65 years and measure the percent body fat of each one.
    
    * The sample mean $\bar{x}$ of the 10 data values is a statistic.
  
  – We do not need to use unknown parameters to compute a statistic.
  
  – We often use a statistic to *estimate* and unknown parameter.
  
  – But the exact value of a particular statistic will be different in different samples drawn from the same population.

  * sampling variability
Randomness

• Chance behavior is unpredictable in the short run but has a predictable pattern in the long run.

• Example: tossing a coin
  – The proportion of heads in a small number of coin tosses is very variable.
  – As more and more tosses are done, the proportion settles down. It gets close to 0.5 and stays there.

• French naturalist Count Buffon (1707-1788) tossed a coin 4040 times and got 2048 heads.
  – proportion heads: $\frac{2048}{4040} = 0.5069$

• While imprisoned by the Germans during World War II, South African mathematician John Kerrich tossed a coin 10,000 times and got 5067 heads.
  – proportion heads: $\frac{5067}{10000} = 0.5067$

• In 1900 English statistician Karl Pearson tossed a coin 24,000 times and got 12,012 heads.
  – proportion heads: $\frac{12012}{24000} = 0.5005$

• American statistician Kate Cowles (19?? - 20??) tossed a coin 5 times and got 4 heads
  – proportion heads: $\frac{4}{5} = 0.8$

• She repeated the experiment and got 2 heads
  – proportion heads: $\frac{2}{5} = 0.4$

The sample space $S$ is the set of all possible outcomes of a random experiment.

Examples:

• We flip a coin and record the outcome as a head or tail

• We draw an 18-year-old American male “at random” and follow up to find out whether he lives to be 65

• We draw an American child at random and record birth order

• A researcher feeds a baby rat a particular diet and records the rat’s weight gain from birth to age 30 days
An **event** is any outcome or set of outcomes of a random experiment.

Example: At random, we draw a child born in the US and record his/her live birth order. We would observe one of the following events:

She is

- 1st child
- 2nd child
- 3rd child
- 4th child
- 5th child
- 6th or later

Or, we might lump certain outcomes together into a single event of interest.

- Child is “1st child” or “not 1st child”

The **probability** of an event is the proportion of times the outcome would occur in a very long series of repetitions under the same conditions.

- (This is the “long-run frequency” definition of probability.)

- coin tosses: the probability of getting a head is 0.5

- birth order of randomly drawn American child

<table>
<thead>
<tr>
<th>Birth order</th>
<th>1st</th>
<th>2nd</th>
<th>3rd</th>
<th>4th</th>
<th>5th</th>
<th>6+</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability</td>
<td>0.416</td>
<td>0.330</td>
<td>0.158</td>
<td>0.058</td>
<td>0.021</td>
<td>0.017</td>
</tr>
</tbody>
</table>

The probability that an event occurs is often denoted with the letter P.

- $P(A)$ is the probability of event $A$

Capital letters near the beginning of the alphabet often are used to denote events.

Example:

- A might represent the event that the child is a 1st child.
- B might represent the event that the child is not a first child.

More probability terminology

- The event (A does not occur) is called the *complement of A* and represented by $A^c$
  - If A is the event that the randomly drawn child is a first-born child, then what is $A^c$?

- Two events A and B that cannot occur simultaneously are *disjoint* or *mutually exclusive*

- The union of two events is the event that one or the other or both occur.
  - The union of events A and B is the event (A or B or both)

- The intersection of two events is the event that both occur.
- The intersection of events A and B is the event (A and B)

• Two events A and B are independent if the probability that one occurs does not change the probability that the other one occurs.

- Example: Suppose one person tosses a penny and another person tosses a dime. The outcomes of the two tosses are independent. Each has a probability of \( \frac{1}{2} \) of being a head. The outcome for one of the coins has no effect on the probabilities of the two possible outcomes for the other coin.

- Example 2: What if the same person tossed the same coin twice?

- Example 3: I have a deck of cards. I draw a card at random. Without putting it back, I draw a second card at random. The event A is that the first card is a heart. The event B is that the second card is a heart. Are events A and B independent?

### Probability models

- mathematical models for randomness!
- consist of two parts
  - a sample space S
  - a way of assigning probabilities to events

### Probability rules

1. Any probability is a number between 0 and 1.

   If \( P(A) \) is the probability of any event A, then
   \[
   0 \leq P(A) \leq 1
   \]

2. All possible outcomes taken together must have probability 1.

   \[
   P(S) = 1
   \]

- One of the possible outcomes has to happen!
3. The probability that an event does not occur is 1 minus the probability that the event does occur.

- \( P(A^c) = 1 - P(A) \)
- Example: If the probability that a randomly selected black American has type O blood is 0.49, what is the probability that he or she has some other blood type?

4. (Addition rule): If two events are mutually exclusive, then the probability that one or the other occurs is the sum of their individual probabilities.

- If A and B are disjoint events, then \( P(A \text{ or } B) = P(A) + P(B) \)
- Example: For our child, we might wish to define an event as
  - “1st or 2nd child” = either “1st child” or “2nd child”
  - Since being a 1st child and being a 2nd child are mutually exclusive events then
    \[
    P(1\text{st or 2nd}) = P(1\text{st}) + P(2\text{nd})
    = 0.416 + 0.330
    = 0.746
    \]

5. This rule can be extended to three or more mutually exclusive events.

- If A, B, and C are all mutually exclusive then
  \[
  P(A \text{ or } B \text{ or } C) = P(A) + P(B) + P(C)
  \]
- Example:
  \[
  P(1\text{st, 2nd, or 3rd}) = P(1\text{st}) + P(2\text{nd}) + P(3\text{rd})
  = 0.416 + 0.330 + 0.158
  = 0.904
  \]

How else might we have computed \( P(1\text{st, 2nd, or 3rd}) \)?

6. (Multiplication rule for independent events): If two events A and B are independent,

\[
P(A \text{ and } B) = P(A) \cdot P(B)
\]

- Example: Suppose I have two separate, complete decks of cards (52 cards in each).
  - I draw one card at random from the first deck. What is the probability that that card is a heart?
  - If I draw one card at random from the first deck and another card at random from the second deck, what is the probability that both cards are hearts?
Assigning probabilities when the sample space is finite

- Assign a probability to each individual outcome.
- These probabilities must all be numbers between 0 and 1, and they must sum to 1.
- Example: Our table of probabilities of the birth positions of American kids is a probability model.

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