Midterm Exam II

Please write your answers in the exam books provided. You can use formulas from page 2 when appropriate.

1. (5 Points) Let $X$ be a continuous random variable with density

$$f_X(x) = \begin{cases} \frac{3}{4}(1 - x^2) & \text{for } -1 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Compute the mean and variance of $X$.

2. (5 Points) Let $X$ and $Y$ be jointly continuous with joint density

$$f(x, y) = \begin{cases} C(x + y)e^{-(x+y)} & \text{for } x \geq 0, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

for some $C > 0$. Find the value of $C$ and the marginal density of $U = (X+Y)/2$. If this is the density of a standard distribution, identify the distribution by name.

3. (5 Points) Let $X$ have a uniform distribution on $[0, 1]$ and let the conditional distribution of $Y$ given $X = x$ be uniform on $[0, x]$. Find the correlation of $X$ and $Y$. [Hint: Use properties of conditional expectations, such as the law of total expectation and the variance decomposition formula, for your calculations. If you do not recall the mean or variance of the Uniform $[0, 1]$ distribution, use the fact that this is a Beta$(1, 1)$ distribution.]

4. (5 Points) Let $N$ have a Poisson distribution with mean $\lambda$ and let the conditional distribution of $X$ given $N = n$ be Binomial$(n, p)$. Compute the moment generating function of the marginal distribution of $X$ and use it to identify this marginal distribution.
### Some Distributions

**Bernoulli(p)**

- pmf: \( P(X = x|p) = p^x(1-p)^{1-x}; \) if \( x = 0, 1; \) \( 0 \leq p \leq 1 \)
- mean, variance: \( E[X] = p, \ Var(X) = p(1-p) \)
- mgf: \( M_X(t) = (1-p) + pe^t \)

**Binomial(n, p)**

- pmf: \( P(X = x|n, p) = \binom{n}{x} p^x(1-p)^{n-x}; \) if \( x = 0, 1, \ldots, n; \) \( 0 \leq p \leq 1 \)
- mean, variance: \( E[X] = np, \ Var(X) = np(1-p) \)
- mgf: \( M_X(t) = ((1-p) + pe^t)^n \)

**Poisson(λ)**

- pmf: \( P(X = x|\lambda) = \frac{\lambda^x e^{-\lambda}}{x!}; \) if \( x = 0, 1, \ldots; \) \( 0 \leq \lambda < \infty \)
- mean, variance: \( E[X] = \lambda, \ Var(X) = \lambda \)
- mgf: \( M_X(t) = e^{\lambda(e^t - 1)} \)

**Geometric(p)**

- pmf: \( P(X = x|p) = p(1-p)^{x-1}; \) if \( x = 1, 2, \ldots; \) \( 0 < p \leq 1 \)
- mean, variance: \( E[X] = \frac{1}{p}, \ Var(X) = \frac{1-p}{p^2} \)
- mgf: \( M_X(t) = \frac{pe^t}{1-(1-p)e^t} \)

**Negative Binomial(r, p)**

- pmf: \( P(X = x|r, p) = \binom{r+x-1}{x} p^r(1-p)^x; \) if \( x = 0, 1, \ldots; \) \( 0 < p \leq 1 \)
- mean, variance: \( E[X] = \frac{r(1-p)}{p}, \ Var(X) = \frac{r(1-p)}{p^2} \)
- mgf: \( M_X(t) = \left(\frac{p}{1-(1-p)e^t}\right)^r \)

**Beta(α, β)**

- pdf: \( f(x|\alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}; \) if \( 0 < x < 1 \)
- mean, variance: \( E[X] = \frac{\alpha}{\alpha+\beta}, \ Var(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} \)

**Cauchy(θ, σ)**

- pdf: \( f(x|\theta, \sigma) = \frac{1}{\pi\sigma} \frac{1}{1+(x-\theta)^2}; \) if \( -\infty < x < \infty; \) \(-\infty < \theta < \infty; \) \( \sigma > 0 \)
- mean, variance: does not exist
- mgf: does not exist

**Gamma(α, β)**

- pdf: \( f(x|\alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1}e^{-x/\beta}; \) if \( 0 < x < \infty; \) \( \alpha, \beta > 0 \)
- mean, variance: \( E[X] = \alpha\beta, \ Var(X) = \alpha\beta^2 \)
- mgf: \( M_X(t) = \left(\frac{1}{1-\beta t}\right)^\alpha, \) if \( t < \frac{1}{\beta} \)

**Normal(μ, σ²)**

- pdf: \( f(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} \exp\{-\frac{1}{2\sigma^2}(x-\mu)^2\}; \) \( \sigma^2 > 0 \)
- mean, variance: \( E[X] = \mu, \ Var(X) = \sigma^2 \)
- mgf: \( M_X(t) = \exp\{\mu t + \frac{1}{2}t^2\sigma^2\} \)
Solutions

1. $E[X] = 0$ since the density is symmetric about the origin. The variance is

$$\text{Var}(X) = E[X^2] = \frac{3}{4} \int_{-1}^{1} x^2 - x^4 dx = \frac{3}{2} \int_{0}^{1} x^2 - x^4 dx$$

$$= \frac{3}{2} \left( \frac{1}{3} - \frac{1}{5} \right) = \frac{3}{2} \times \frac{2}{15} = \frac{1}{5} = 0.2.$$ 

2. The integral of the joint density is

$$\int_{0}^{\infty} \int_{0}^{\infty} C(x+y)e^{-(x+y)}dxdy = C \left[ \int_{0}^{\infty} \int_{0}^{\infty} xe^{-x-y}dxdy + \int_{0}^{\infty} \int_{0}^{\infty} ye^{-x-y}dxdy \right]$$

$$= C \left[ \int_{0}^{\infty} xe^{-x}dx \int_{0}^{\infty} e^{-y}dy + \int_{0}^{\infty} e^{-x}dx \int_{0}^{\infty} ye^{-y}dy \right]$$

$$= C \left[ 1 + 1 \right] = 2C$$

So $C = 1/2$.

Let $V = Y$. The transformation from $X, Y$ to $U, V$ is one-to-one with inverse

$$x = 2u - v$$

$$y = v$$

and Jacobian determinant

$$\det J(u, v) = \det \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix} = 2.$$ 

The range of the transformation is

$$\mathcal{B} = \{(u, v) : 0 \leq v \leq 2u \}.$$ 

The joint density of $U$ and $V$ is

$$f_{U,V}(u,v) = f_{X,Y}(2u - v, v) |J(u, v)| = \begin{cases} 2ue^{-2u} & \text{for } 0 \leq v \leq 2u \\ 0 & \text{otherwise.} \end{cases}$$

The marginal density of $U$ is therefore

$$f_U(u) = \int_{0}^{2u} 2ue^{-2u}dv = ue^{-2u} \times 2u = 4u^2e^{-2u}$$

for $u \geq 0$ and zero otherwise. This is the density of a Gamma($\alpha = 3, \beta = 1/2$) distribution.
3. The mean and variance of $X$ are

$$E[X] = \frac{1}{2}, \quad \text{Var}(X) = \frac{1}{12}. $$

The mean of $Y$ is

$$E[Y] = E[E[Y|X]] = E[X/2] = E[X]/2 = \frac{1}{4} $$

and the variance of $Y$ is

$$\text{Var}(Y) = E[\text{Var}(Y|X)] + \text{Var}(E[Y|X])$$
$$= E[X^2/12] + \text{Var}(X/2) = E[X^2]/12 + \text{Var}(X)/4$$
$$= \frac{1}{3} \times \frac{1}{12} + \frac{1}{12} \times \frac{1}{4} = \frac{1}{12} \left( \frac{1}{3} + \frac{1}{4} \right) = \frac{1}{12} \times \frac{7}{12} = \frac{7}{144}. $$

The expected product is

$$E[XY] = E[E[XY|X]] = E[XE[Y|X]] = E[X(X/2)] = E[X^2]/2 = \frac{1}{6}. $$

The covariance is

$$\text{Cov}(X,Y) = E[XY] - E[X]E[Y] = \frac{1}{6} - \frac{1}{2} \times \frac{1}{4} = \frac{1}{6} - \frac{1}{8} = \frac{1}{24}. $$

The correlation is therefore

$$\rho = \frac{\text{Cov}(X,Y)}{\sigma_x \sigma_y} = \frac{1/24}{\sqrt{(1/12)(7/144)}} = \sqrt{\frac{3}{7}} \approx 0.6547. $$

4. The moment generating function of $X$ is

$$M_X(t) = E[e^{tX}] = E[E[e^{tX}|N]]. $$

The moment generating function of the conditional distribution of $X$ given $N = n$ is

$$E[e^{tX}|N = n] = ((1 - p) + pe^t)^n,$$

so

$$E[e^{tX}|N] = ((1 - p) + pe^t)^N,$$

and

$$M_X(t) = E[((1 - p) + pe^t)^N] = E[\exp\{N \log((1 - p) + pe^t)\}] = M_N(\log((1 - p) + pe^t)) = \exp\{\lambda((1 - p) + pe^t - 1)\} = \exp\{\lambda(p(e^t - 1))\}. $$

This is the moment generating function of a Poisson distribution with mean $\lambda p$. 

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Summary of Scores

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Histogram of m2

Frequency

m2

10 12 14 16 18 20

0 2 4 6 8