Sample Midterm Exam II

Please write your answers in the exam books provided. You can use formulas from page 2 when appropriate.

1. (5 Points) Let $X$, $Y$, and $Z$ be independent random variables with means and variances

$$
\begin{align*}
\mu_X &= 0 \\
\mu_Y &= 2 \\
\mu_Z &= 3 \\
\sigma_X^2 &= 5 \\
\sigma_Y^2 &= 5 \\
\sigma_Z^2 &= 9
\end{align*}
$$

Let $U = XZ$ and $V = YZ$. Find the covariance of $U$ and $V$.

2. (5 Points) Let $X$ have density $f(x) = \frac{1}{2}e^{-|x|}$. Find the mean and variance of $X$.

3. (5 Points) Suppose $X$ and $Y$ are independent continuous non-negative random variables with densities

$$
\begin{align*}
f_X(x) &= \begin{cases} 2xe^{-x^2} & \text{for } x > 0 \\ 0 & \text{otherwise} \end{cases} \quad f_Y(y) = \begin{cases} e^{-y} & \text{for } y > 0 \\ 0 & \text{otherwise} \end{cases}
\end{align*}
$$

Find the density of $U = X^2/Y$.

4. (5 Points) Let $P$ have a distribution on $[0,1]$ with density $f(p) = 3p^2$. The conditional distribution of $X$ given $P = p$ is Geometric($p$), with possible values $1, 2, \ldots$. Find the mean and variance of $X$. 


### Some Distributions

<table>
<thead>
<tr>
<th>Distribution</th>
<th>pmf</th>
<th>mean, variance</th>
<th>mgf</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Bernoulli</strong></td>
<td>$P(X = x</td>
<td>p) = p^x(1 - p)^{1-x}; x = 0, 1; 0 \leq p \leq 1$</td>
<td>$E[X] = p$, $Var(X) = p(1-p)$</td>
</tr>
<tr>
<td><strong>Binomial</strong></td>
<td>$P(X = x</td>
<td>n, p) = \binom{n}{x} p^x(1 - p)^{n-x}; x = 0, 1, \ldots, n; 0 \leq p \leq 1$</td>
<td>$E[X] = np$, $Var(X) = np(1-p)$</td>
</tr>
<tr>
<td><strong>Poisson</strong></td>
<td>$P(X = x</td>
<td>\lambda) = \frac{e^{-\lambda} \lambda^x}{x!}; x = 0, 1, \ldots; 0 \leq \lambda &lt; \infty$</td>
<td>$E[X] = \lambda$, $Var(X) = \lambda$</td>
</tr>
<tr>
<td><strong>Geometric</strong></td>
<td>$P(X = x</td>
<td>p) = p(1 - p)^{x-1}; x = 1, 2, \ldots; 0 &lt; p \leq 1$</td>
<td>$E[X] = \frac{1}{p}$, $Var(X) = \frac{1-p}{p^2}$</td>
</tr>
<tr>
<td><strong>Negative Binomial</strong></td>
<td>$P(X = x</td>
<td>r, p) = \binom{r+x-1}{x} p^r(1-p)^x; x = 0, 1, \ldots; 0 &lt; p \leq 1$</td>
<td>$E[X] = \frac{r(1-p)}{p}$, $Var(X) = \frac{r(1-p)}{p^2}$</td>
</tr>
<tr>
<td><strong>Beta</strong></td>
<td>$f(x</td>
<td>\alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}; 0 &lt; x &lt; 1$</td>
<td>$E[X] = \frac{\alpha}{\alpha+\beta}$, $Var(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$</td>
</tr>
<tr>
<td><strong>Cauchy</strong></td>
<td>$f(x</td>
<td>\theta, \sigma) = \frac{1}{\pi\sigma} \frac{1}{1+(\frac{x-\theta}{\sigma})^2}; -\infty &lt; x &lt; \infty; -\infty &lt; \theta &lt; \infty; \sigma &gt; 0$</td>
<td>mean, variance does not exist</td>
</tr>
<tr>
<td><strong>Gamma</strong></td>
<td>$f(x</td>
<td>\alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1}e^{-x/\beta}; 0 &lt; x &lt; \infty; \alpha, \beta &gt; 0$</td>
<td>$E[X] = \alpha\beta$, $Var(X) = \alpha\beta^2$</td>
</tr>
<tr>
<td><strong>Normal</strong></td>
<td>$f(x</td>
<td>\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma}} \exp{-\frac{1}{2\sigma^2}(x - \mu)^2}; \sigma^2 &gt; 0$</td>
<td>$E[X] = \mu$, $Var(X) = \sigma^2$</td>
</tr>
</tbody>
</table>
Solutions

1. Since $X$, $Y$, and $Z$ are independent,
   \[ E[V] = E[YZ] = E[Y]E[Z] = 2 \times 3 = 6 \]


2. The density is symmetric about zero, so $E[X] = 0$, and the variance is
   \[ Var(X) = E[X^2] = \int_{-\infty}^{\infty} x^2 \frac{1}{2} e^{-|x|} dx = \int_{0}^{\infty} x^2 e^{-x} dx = \Gamma(3) = 2! = 2. \]

3. Let $V = Y$. Then the transformation from $X, Y$ to $U, V$ is one to one with inverse $x = \sqrt{uv}$ and $y = v$. The Jacobian determinant of the inverse is
   \[
   \det \begin{bmatrix}
   \frac{1}{2} \sqrt{\frac{u}{v}} & \frac{1}{2} \sqrt{\frac{v}{u}} \\
   0 & 1
   \end{bmatrix} = \frac{1}{2} \sqrt{\frac{v}{u}}.
   \]

   The range of the transformation is $B = (0, \infty) \times (0, \infty)$. The joint density of $U, V$ is
   \[ f_{U,V}(u, v) = 2 \sqrt{uv} e^{-uv} e^{-v} \frac{1}{2} \sqrt{\frac{v}{u}} = ve^{-(1+u)v} \]
   for $u, v > 0$. The marginal density of $U$ is therefore
   \[ f_U(u) = \int_{0}^{\infty} ve^{-(1+u)v} dv = \frac{1}{(1 + u)^2} \]
   for $u > 0$.

4. The conditional mean and variance of $X$ given $P$ are $E[X|P] = 1/P$ and $Var(X|N, P) = (1 - P)/P^2$. The mean of $X$ is
   \[ E[X] = E[E[X|N]] = E[1/P] = \int_{0}^{1} \frac{1}{p} 3p^2 dp = \int_{0}^{1} 3dp = \frac{3}{2}. \]

   The variance of $X$ is
   \[ Var(X) = E[Var(X|P)] + Var(E[X|P]) = E[(1 - P)/P^2] + Var(1/P) \]
   \[ = 2 \int_{0}^{1} \frac{1}{p^2} 3p^2 dp - \frac{3}{2} - \frac{9}{4} = 2 \int_{0}^{1} 3dp - \frac{15}{4} = 6 - \frac{15}{4} = \frac{9}{4}. \]