probability of 2 defects in 2 square yards is

$$\begin{aligned}
&P[(X_1 = 1 \text{ and } X_2 = 1) \text{ or } (X_1 = 2 \text{ and } X_2 = 0) \text{ or } (X_1 = 0 \text{ and } X_2 = 2)] \\
&\overset{m.e.}{=} P(X_1 = 1 \text{ and } X_2 = 1) + P(X_1 = 2 \text{ and } X_2 = 0) + P(X_1 = 0 \text{ and } X_2 = 2) \\
&\overset{ind}{=} P(X_1 = 1)P(X_2 = 1) + P(X_1 = 2)P(X_2 = 0) + P(X_1 = 0)P(X_2 = 2) \\
&= \frac{e^{-1.2}1.2^1}{1!}\frac{e^{-1.2}1.2^1}{1!} + \frac{e^{-1.2}1.2^2}{2!}\frac{e^{-1.2}1.2^0}{0!} + \frac{e^{-1.2}1.2^0}{0!}\frac{e^{-1.2}1.2^2}{2!} \\
&= 0.2613
\end{aligned}$$

Obviously, this is a lot more work than simply "scaling" the Poisson distribution as in part (c). Both methods are mathematically equivalent, however.

(e) Suppose we examine 100 pieces of cloth; each piece is 1 square yard in size. Determine the probability that exactly 15 of the pieces contain 2 defects.

The number of defects in 1 square yard, X, has a Poisson distribution with mean $\mu = E(X) = \lambda = 1.2$, i.e. $X \sim Pois(\lambda = 1.2)$. The probability of 2 defects in 1 square yard is

$$P(X = 2) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-1.2}1.2^2}{2!} = 0.21686$$

If we have 100 pieces of cloth, we essentially have 100 independent trials, each trail results in a success or failure (where a success is having 2 defects, and a failure is not having 2 defects), and the probability of success (having 2 defects) is 0.21686. If the random variable Y equals the number of pieces of cloth with 2 defects, then

$$Y \sim Bin(n = 100, p = 0.21686)$$

Therefore, the probability that exactly 15 of the pieces of cloth have 2 defects is

$$P(Y = 15) = \binom{n}{y}p^y(1-p)^{n-y} = \binom{100}{15}0.21686^{15}(1-0.21686)^{100-15} = 0.0265$$

4.6 Exercises

♥ = answers are provided beginning on page 229.

4.1 ♥ Suppose the random variable X has probability distribution

x:	0	1	2	3	4
$P(X = x)$:	0.2	0.1	0.2	0.2	0.3

Find the following probabilities.

(a) $P(X \leq 2)$

(b) $P(X < 2)$

(c) $P(X \leq 2 \text{ and } X < 4)$

(d) $P(X \leq 2 \text{ and } X \geq 1)$

(e) $P(X = 1 \text{ or } X \geq 3)$

(f) $P(X = 1 \text{ or } X < 3)$

(g) $P(X = 2|X \le 2)$

(h) $P(X \le 2|X \ge 1)$

4.2 ♥ Suppose a 4-sided die is rolled twice (the sides of the die are labeled 1, 2, 3, and 4). Let the random variable X equal the *sum* of the two rolls.

(a) Find the probability distribution of X.

(b) Find the probability that the sum of the two rolls is less than or equal to 3, i.e. find $P(X \le 3)$.

(c) Find the probability that the sum of the two rolls is less than 3 *or* greater than or equal to 6, i.e. find $P(X < 3 \text{ or } X \ge 6)$.

(d) Find the probability that the sum of the two rolls is greater than 2 *or* less than or equal to 7, i.e. find $P(X > 2 \text{ or } X \le 7)$.

(e) Find $P(X \ge 4 \text{ and } X < 6)$.

(f) Find the probability that the sum of the two rolls is greater than 4 given that the sum is greater than 2, i.e. find $P(X > 4|X > 2)$.

(g) What is the sum of the 2 rolls on average?

(h) Compute $SD(X)$.

4.3 A basket contains 4 puppies: one of the puppies has 1 spot, one of the puppies has 2 spots, and the remaining two puppies have 4 spots. Suppose *two* puppies are selected at random *without* replacement. Let the random variable X equal the *total* number of spots on the selected puppies.

(a) Find the probability distribution of X.

(b) Find the probability that the puppies have a total of 5 spots, i.e. find $P(X = 5)$.

(c) Find the probability that the puppies have a total of 6 or more spots, i.e. find $P(X \ge 6)$.

(d) Find the probability that the puppies have 5 or fewer spots *or* 8 spots, i.e. find $P(X \le 5 \text{ or } X = 8)$.

(e) Given that the puppies have 6 or more spots, determine the probability that both puppies have 4 spots each (i.e. 8 spots total), i.e. find $P(X = 8|X \ge 6)$.

(f) On average, how many spots do we expect on the two selected puppies?

(g) Compute $\sigma^2 = Var(X)$.

4.4 A large warehouse contains 2-packs, 4-packs, and 8-packs of batteries. Suppose the random variable X equals the number of batteries in a randomly selected package of batteries. It is known that X has probability distribution

$$P(X = x) = \frac{8}{7x} \quad \text{for } x = 2, 4, 8$$

(a) What is $P(X = 2)$?

(b) Determine $P(X \ge 4)$.

(c) Find $P(X = 2 \text{ or } X = 8)$.

(d) Find $\mu = E(X)$.

4.5 Suppose the discrete random variable X has probability distribution

$$P(X = x) = \frac{1}{2^x} \quad \text{for } x = 1, 2, \ldots$$

(a) Find $P(X = 5)$.

(b) Determine $P(X \geq 2)$.

(c) Find $P(X \leq 4 \text{ and } X \geq 4)$.

(d) Find $P(X \leq 4 \text{ or } X \geq 4)$.

(e) Find $P(X \geq 10 \text{ or } X \geq 2)$.

(f) Determine $P(X \leq 3 | X \geq 2)$.

4.6 A street vendor is asking people to play a simple game. You roll a pair of dice. If the sum on the dice is 10 or higher, you win \$10. If you roll a pair of 1's, you win \$50. Otherwise you lose \$5. If the random variable X equals your win (or loss) for each play, find $\mu = E(X)$ (i.e. figure out how much we expect to win or lose for each play, on average). Is it wise to play this game? Why?

4.7 ♥ Suppose a bowl has 5 chips; two chips are labeled "2", and three chips are labeled "3". Suppose *two* chips are selected at random *with* replacement. Let the random variable X equal the *product* of the two draws (e.g. if the first draw is a 2 (2_1) and the second draw is a 3 (3_2), then the product is $2 \times 3 = 6$).

(a) Find the probability distribution of X.

(b) Find the probability that the *product* of the two draws is less than or equal to 6, i.e. find $P(X \leq 6)$.

(c) Find the probability that the *product* of the two draws is greater than 4 given that the product is less than or equal to 6, i.e. find $P(X > 4 | X \leq 6)$.

(d) Compute the expected value of X.

(e) Compute $\sigma = SD(X)$.

4.8 Repeat question 4.7(a) assuming the chips are drawn *without* replacement.

4.9 Suppose a bowl has 9 chips; one chip is labeled "1", three chips are labeled "3", and five chips are labeled "5". Suppose *two* chips are selected at random *with* replacement. Let the random variable X equal the *absolute difference* between the two draws (e.g. if the first draw is a 1 (1_1) and the second draw is a 5 (5_2), then the absolute difference is $|1 - 5| = 4$).

(a) Find the probability distribution of X.

(b) Find the probability that both draws are the same.

(c) Find the probability that both draws are *not* the same.

(d) Given that both draws are *not* the same, determine the probability that the absolute difference is equal to 2, i.e. find $P(X = 2 | X > 0)$.

(e) On average, what is X equal to?

4.10 Repeat question 4.9(a) assuming the chips are drawn *without* replacement.

4.11 ♥ Suppose the random variable X has the following probability distribution.

$x:$	1	2	4
$P(X = x):$	0.1	0.8	0.1

Find the mean $\mu = E(X)$, standard deviation $\sigma = SD(X)$, and variance $\sigma^2 = Var(X)$ of the random variable X.

4.12 Suppose the random variable Y has the following probability distribution.

$y:$	10	20	40
$P(Y = y):$	0.1	0.8	0.1

(a) Find the mean $\mu = E(Y)$, standard deviation $\sigma = SD(Y)$, and variance $\sigma^2 = Var(Y)$ of the random variable Y.

(b) In general, for any constant c, $E(cX) = cE(X)$. Looking back at question 4.11, does this rule hold? Why?

(c) In general, for any constant c, $Var(cX) = c^2 Var(X)$. Looking back at question 4.11, does this rule hold? Why?

4.13 It is known that 15% of US home mortgages are under water (i.e. the homeowner owes more than the house is worth). Suppose 18 mortgages are randomly selected (assume independence). Let the random variable X equal the number that are under water.

(a) What is the distribution of X? *Be sure to state all parameters.*

(b) Find the probability that exactly 3 are under water.

(c) Find the probability that 1 or more are under water.

(d) Given that 1 or more are under water, determine the probability that 2 or more are underwater.

(e) On average, how many do we expect to be under water?

(f) Find $Var(X)$.

4.14 In reference to question (4.13), suppose mortgages are repeatedly selected at random (assume independence).

(a) Suppose the random variable X is the mortgage that is the first to be under water. What is the distribution of X? *Be sure to state the parameter.*

(b) Find the probability that the 10th selected mortgage is the first that is under water.

(c) Find the probability that the first under water mortgage occurs on or before the 3rd selected, i.e. find $P(X \leq 3)$.

(d) Find the probability that the first under water mortgage occurs after the 2nd selected, i.e. find $P(X > 2)$.

(e) Given that the first mortgage is not under water, find the probability that the second mortgage is the first to be under water, i.e. find $P(X = 2|X \geq 2)$.

(f) On average, how many mortgages must be selected to get the first under water?

4.15 ♥ It is known that 20% of all credit applicants have poor credit ratings. Suppose 30 applicants are randomly selected (assume independence). Let the random variable X equal the number of applicants with poor credit ratings.

(a) What is the distribution of X? *Be sure to state all parameters.*

(b) Find the probability that exactly 8 have poor credit.

(c) Find $P(8 \le X < 11)$.

(d) On average, how many do we expect to have poor credit?

(e) Find $SD(X)$.

(f) Use the applet at

`http://www.stat.uiowa.edu/~mbognar/applets/bin.html`

to find the probability that 10 or fewer have poor credit.

(g) Use the applet to find the probability that 7 or more have poor credit. *Hint: Use the complement rule.*

4.16 An egg manufacturer knows that 9.6% of its eggs are cracked. The eggs are packed in cartons containing 12 eggs. *Assume eggs are independent.*

(a) If the random variable X counts the total number of cracked eggs in a carton, determine the distribution of X and find $\mu = E(X)$ and $\sigma = SD(X)$.

(b) Suppose a carton of eggs is randomly selected. Find the probability that exactly 3 eggs are cracked.

(c) Suppose a carton of eggs is randomly selected. Find the probability that 11 or fewer eggs are cracked.

(d) Suppose we randomly select 10 cartons of eggs (assume independence). Determine the probability that all 10 cartons have 0 broken eggs. *Hint: First find the probability that a carton contains 0 broken eggs, then find the probability.*

(e) Suppose we randomly select 10 cartons of eggs (assume independence). Determine the probability that exactly 7 of the cartons have 0 broken eggs. *Hint: First find the probability that a carton contains 0 broken eggs, then make use of the binomial distribution.*

(f) Suppose eggs are repeatedly selected at random. Find the probability that the 10^{th} selected egg is the 1^{st} cracked egg.

(g) Suppose eggs are repeatedly selected at random. Find the probability that the 10^{th} selected egg is the 2^{nd} cracked egg. *This one is a little more challenging.*

4.17 ♥ Suppose 2 cars cross a bridge per minute, on average. Assume the assumptions for the Poisson distribution are met.

(a) Suppose the random variable X equals the number of cars that cross the bridge in 1 minute. Determine the distribution of X and find $\mu = E(X)$ and $\sigma^2 = Var(X)$.

(b) Find the probability that exactly 3 cars cross the bridge in 1 minute.

(c) Find the probability that 1 or more cars cross the bridge in 1 minute.

(d) Suppose Y equals the number of cars that cross the bridge in 2 minutes. Determine the distribution of Y and find $\mu = E(Y)$ and $\sigma = SD(Y)$.

(e) Find the probability that exactly 4 cars cross the bridge in 2 minutes.

(f) Suppose a worker observes the bridge for 2 minutes. Find the probability that 3 cars cross in the first minute and 1 car crosses in the second minute. *Hint: Remember that the number of occurrences in non-overlapping intervals in a Poisson process are independent. If we let X_1 and X_2 equal the number of crossings in the first and second minute, respectively, then we can find $P(X_1 = 3 \text{ and } X_2 = 1)$.*

(g) Suppose we observe 30 one-minute time intervals. Determine the probability that there are 0 car crossings in exactly 5 of the intervals. *Hint: First determine the probability of 0 car crossings in 1 minute, then make use of the binomial distribution.*

4.18 The number of emails per day in Matt's inbox can be modeled by a Poisson random variable with mean 30.

(a) Suppose the random variable X equals the number of emails in 1 hour. Determine the distribution of X and find $E(X)$ and $SD(X)$.

(b) What is the probability that Matt receives 3 emails in one hour?

(c) What is the probability that Matt receives 1 or more emails in one hour?

(d) Suppose the random variable X equals the number of emails in 2 days. What is the distribution of X? Use the applet at

`http://www.stat.uiowa.edu/~mbognar/applets/pois.html`

to find $P(X \leq 50)$.

(e) Suppose Matt's inbox is watched over a 2 day period. Find the probability that 20 emails arrive in the first day and 30 emails arrive in the second day. *Hint: Remember that the number of occurrences in non-overlapping intervals in a Poisson process are independent. If we let X_1 and X_2 equal the number of emails in the first and second day, respectively, then we can find $P(X_1 = 20 \text{ and } X_2 = 30)$.*

(f) Suppose Matt's inbox is watched over a 3 day period. Find the probability that 20 emails arrive in the first day and 50 emails arrive in the last 2 days. *Hint: If we let X_1 equal the number of emails in the first day, and let X_2 equal the number of emails in the last 2 days, then we can find $P(X_1 = 20 \text{ and } X_2 = 50)$.*

(g) Suppose we observe 24 one-hour time intervals. Determine the probability that Matt receives exactly 1 email in 3 of the intervals. *Hint: First determine the probability of 1 email in 1 hour, then make use of the binomial distribution.*

4.19 The average number of defects per square foot of photographic paper is 0.3. Assume the Poisson assumptions hold.

(a) Suppose the random variable X equals the number of defects in 1 square foot of paper. Determine the distribution of X and find $E(X)$ and $SD(X)$.

(b) What is the probability of 1 defect in 1 square foot of paper?

(c) What is the probability of 1 or more defects in 1 square foot of paper?

(d) What is the probability of 2 defects in 2 square feet of paper?

(e) Suppose we have 2 sheets of paper; each sheet is 1 square foot in size. Determine the probability that the first sheet contains 0 defects and the second sheet contains 3 defects. *Hint: Remember that the number of occurrences in non-overlapping regions in a Poisson process are independent. If we let X_1 and X_2 equal the number defects in the first sheet and second sheet, respectively, then we can find* $P(X_1 = 0 \text{ and } X_2 = 3)$.

(f) Suppose we examine 100 sheets of photographic paper; each sheet is 1 square foot in size. Determine the probability that exactly 80 of the sheets contain 0 defects. *Hint: First determine the probability of 0 defects in 1 sheet, then make use of the binomial distribution.*

(g) Suppose we examine 100 sheets of photographic paper; each sheet is 0.25 square feet in size. Determine the probability that exactly 95 of the sheets contain 0 defects. *Hint: First determine the probability of 0 defects in 1 sheet, then make use of the binomial distribution.*

(h) In reference to Exercise 4.19(g), use the applet at

`http://www.stat.uiowa.edu/~mbognar/applets/bin.html`

to find the probability that 95 or fewer sheets have 0 defects.